Question 1 [10]: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to state the row operations used at each step and clearly state the final solution.

$$x - 2y - z - 3w = -3$$
$$-x + y + z = 2$$
$$4y + 3z - 6w = -2$$

$$\begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$$

$R_2 = r_1 + r_2$: $\begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & -1 & 0 & -3 & -1 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$

$R_2 = (-1)r_2$: $\begin{bmatrix} 1 & -2 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 4 & 3 & -6 & -2 \end{bmatrix}$

$R_1 = 2r_2 + r_1$:
$R_3 = (-4)r_2 + r_3$: $\begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 3 & -18 & -6 \end{bmatrix}$

$R_3 = \frac{1}{3}$: $\begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{bmatrix}$

$R_1 = r_3 + r_1$: $\begin{bmatrix} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{bmatrix}$

Let $w = t$

So $z = -2 + 6t$

$y = 1 - 3t$

$x = -3 + 3t$

where t is any real number.

Question 2 [10]: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to state the row operations used at each step and clearly state the final solution.

$$-2x + 6y - z = -10$$
$$x - 2y + 4z = 6$$
$$x + y + 13z = 6$$

$$\begin{bmatrix} -2 & 6 & -1 & -10 \\ 1 & -2 & 4 & 6 \\ 1 & 1 & 13 & 6 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ -2 & 6 & -1 & -10 \\ 1 & 1 & 13 & 6 \end{bmatrix}$

$R_2 = 2r_1 + r_2$:
$R_3 = (-1)r_2 + r_3$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 2 & 7 & 2 \\ 0 & 3 & 9 & 0 \end{bmatrix}$

$r_2 \leftrightarrow r_3$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 3 & 9 & 0 \\ 0 & 2 & 7 & 2 \end{bmatrix}$

$R_2 = \frac{1}{3} r_2$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 7 & 2 \end{bmatrix}$

$R_1 = 2r_2 + r_1$:
$R_3 = -2r_2 + r_3$: $\begin{bmatrix} 1 & 0 & 10 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$R_1 = -10r_3 + r_1$;
$R_2 = -3r_3 + r_2$;

$$\begin{bmatrix} 1 & 0 & 0 & -14 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{\begin{aligned} \therefore\ x &= -14 \\ y &= -6 \\ z &= 2 \end{aligned}}$$

Question 3: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 2 & -2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

(a)[3] Compute $(3\mathbf{A} - 4\mathbf{C})\mathbf{D}$

$$\left(\begin{bmatrix} 3 & 0 \\ 6 & 12 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -16 \\ 4 & 0 \\ 8 & -8 \end{bmatrix}\right)\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 & 16 \\ 2 & 12 \\ -11 & 14 \end{bmatrix}\begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -70 \\ 0 \\ -80 \end{bmatrix}$$

(b)[3] Compute $(\mathbf{AB} + 3\mathbf{I}_3)^T$

$$AB + 3I_3 = \begin{bmatrix} 4 & -3 & 0 \\ 12 & -2 & -8 \\ -2 & 5 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 0 \\ 12 & 1 & -8 \\ -2 & 5 & -1 \end{bmatrix}$$

$$\therefore (AB + 3I_3)^T = \begin{bmatrix} 7 & 12 & -2 \\ -3 & 1 & 5 \\ 0 & -8 & -1 \end{bmatrix}$$

(c)[2] Compute $\operatorname{tr}(\mathbf{AB} - \mathbf{BA})$

Not defined since AB is 3×3 but BA is 2×3.

(d)[2] Suppose there is some matrix $\mathbf{P}$ such that the product $\mathbf{APC}$ is defined. What must be the size of the matrix $\mathbf{P}$?

$$\begin{array}{ccc} A & P & C \\ 3 \times 2 & 2 \times 3 & 3 \times 2 \end{array}$$

$\therefore$ P must be 2×3.

Question 4:

(a)[7] Determine $\mathbf{A}^{-1}$ where $\mathbf{A}$ is the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_1 = r_2 + r_1$:
$R_3 = (-1)r_2 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

$R_3 = (-1)r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$R_1 = (-3)r_3 + r_1$;
$R_2 = (-3)r_3 + r_2$;
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -2 & 3 \\ 0 & 1 & 0 & -5 & -2 & 3 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -4 & -2 & 3 \\ -5 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix}.$$

(b)[3] Write the following system as a matrix product $\mathbf{Ax} = \mathbf{b}$ and use your result in part (a) to solve the system:

$$\begin{aligned} x - y &= 3 \\ -x + 2y + 3z &= 1 \\ x + 2z &= -7 \end{aligned}$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$= \begin{bmatrix} -4 & -2 & 3 \\ -5 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -35 \\ -38 \\ 14 \end{bmatrix}$$

$\boxed{\therefore x = -35,\ y = -38,\ z = 14}$

Question 5:

(a)[7] Determine all values of c for which the following matrix is invertible:

$$\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} c & c & c & 1 & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$$

$R_1 = \frac{1}{c} r_1$: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$

$R_2 = (-1)r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & c-1 & c-1 & -\frac{1}{c} & 1 & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$

$R_2 = \left(\frac{1}{c-1}\right) r_2$: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$

$R_1 = -r_2 + r_1$: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c} + \frac{1}{c(c-1)} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 1 & -\frac{1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & c-1 & \frac{-1}{c} & 0 & 1 \end{array}\right]$

$R_3 = \left(\frac{1}{c-1}\right) r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 1 & \frac{-1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & 1 & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

$R_2 = (-1) r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & \frac{-1}{c-1} \\ 0 & 0 & 1 & \frac{-1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

$\boxed{\therefore\ c \neq 0,\ c \neq 1}$

(b)[3] Find two non-zero matrices **A** and **B** such that $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.