Question 1 [10]: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to state the row operations used at each step and clearly state the final solution.

$$-2x + 6y - z = -10$$
$$x - 2y + 4z = 6$$
$$x + y + 13z = 6$$

$$\begin{bmatrix} -2 & 6 & -1 & -10 \\ 1 & -2 & 4 & 6 \\ 1 & 1 & 13 & 6 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ -2 & 6 & -1 & -10 \\ 1 & 1 & 13 & 6 \end{bmatrix}$

$R_2 = 2r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 2 & 7 & 2 \\ 0 & 3 & 9 & 0 \end{bmatrix}$

$r_2 \leftrightarrow r_3$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 3 & 9 & 0 \\ 0 & 2 & 7 & 2 \end{bmatrix}$

$R_2 = \frac{1}{3} r_2$: $\begin{bmatrix} 1 & -2 & 4 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 7 & 2 \end{bmatrix}$

$R_1 = 2r_2 + r_1$:
$R_3 = -2r_2 + r_3$: $\begin{bmatrix} 1 & 0 & 10 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$R_1 = -10r_3 + r_1$:
$R_2 = -3r_3 + r_2$:

$$\begin{bmatrix} 1 & 0 & 0 & -14 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\boxed{\therefore\ x = -14,\ y = -6,\ z = 2}$

Question 2 [10]: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to state the row operations used at each step and clearly state the final solution.

$$x - 2y - z - 3w = -3$$
$$-x + y + z = 2$$
$$4y + 3z - 6w = -2$$

$$\left[\begin{array}{ccccc} 1 & -2 & -1 & -3 & -3 \\ -1 & 1 & 1 & 0 & 2 \\ 0 & 4 & 3 & -6 & -2 \end{array}\right]$$

$R_2 = r_1 + r_2$:

$$\left[\begin{array}{ccccc} 1 & -2 & -1 & -3 & -3 \\ 0 & -1 & 0 & -3 & -1 \\ 0 & 4 & 3 & -6 & -2 \end{array}\right]$$

$R_2 = (-1)r_2$:

$$\left[\begin{array}{ccccc} 1 & -2 & -1 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 4 & 3 & -6 & -2 \end{array}\right]$$

$R_1 = 2r_2 + r_1$:
$R_3 = (-4)r_2 + r_3$:

$$\left[\begin{array}{ccccc} 1 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 3 & -18 & -6 \end{array}\right]$$

$R_3 = \frac{1}{3} r_3$:

$$\left[\begin{array}{ccccc} 1 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{array}\right]$$

$R_1 = r_3 + r_1$:

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -6 & -2 \end{array}\right]$$

Let $w = t$,

so $z = -2 + 6t$

$y = 1 - 3t$

$x = -3 + 3t$

where t is any real number.

Question 3: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ -1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 2 & -2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

(a)[3] Compute $(3\mathbf{A} - 4\mathbf{C})\mathbf{D}$

$$\left(\begin{bmatrix} 3 & 0 \\ 6 & 12 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -16 \\ 4 & 0 \\ 8 & -8 \end{bmatrix}\right)\begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 & 16 \\ 2 & 12 \\ -11 & 14 \end{bmatrix}\begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} 70 \\ 0 \\ 80 \end{bmatrix}.$$

(b)[3] Compute $(\mathbf{AB} - 3\mathbf{I}_3)^T$

$$AB - 3I_3 = \begin{bmatrix} 4 & -3 & 0 \\ 12 & -2 & -8 \\ -2 & 5 & -4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 12 & -5 & -8 \\ -2 & 5 & -7 \end{bmatrix}$$

$$\therefore (AB - 3I_3)^T = \begin{bmatrix} 1 & 12 & -2 \\ -3 & -5 & 5 \\ 0 & -8 & -7 \end{bmatrix}.$$

(c)[2] Compute $\mathrm{tr}(\mathbf{AB} - \mathbf{BA})$

Not defined since AB is 3×3 while BA is 2×2.

(c)[2] Suppose there is some matrix **P** such that the product **APC** is defined. What must be the size of the matrix **P**?

$$\begin{array}{ccc} A & P & C \\ \uparrow & \uparrow & \uparrow \\ 3 \times 2 & 2 \times 3 & 3 \times 2 \end{array}$$

$\therefore$ P must be 2×3.

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Question 4:

(a)[7] Determine $\mathbf{A}^{-1}$ where $\mathbf{A}$ is the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = r_1 + r_2$; $R_3 = (-1)r_1 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_1 = r_2 + r_1$; $R_3 = (-1)r_2 + r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

$R_3 = (-1)r_3$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$R_1 = (-3)r_3 + r_1$; $R_2 = (-3)r_3 + r_2$;
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -2 & 3 \\ 0 & 1 & 0 & -5 & -2 & 3 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -4 & -2 & 3 \\ -5 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix}.$$

(b)[3] Write the following system as a matrix product $\mathbf{Ax} = \mathbf{b}$ and use your result in part (a) to solve the system:

$$\begin{aligned} x - y &= -3 \\ -x + 2y + 3z &= -1 \\ x + 2z &= 7 \end{aligned}$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$= \begin{bmatrix} -4 & -2 & 3 \\ -5 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 35 \\ 38 \\ -14 \end{bmatrix} \quad \therefore \; x = 35,\; y = 38,\; z = -14$$

Question 5 [10]:

(a)[7] Determine all values of c for which the following matrix is invertible:

$$\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} c & c & c & 1 & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$$

$R_1 = \frac{1}{c} r_1$: $$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 1 & c & c & 0 & 1 & 0 \\ 1 & 1 & c & 0 & 0 & 1 \end{array}\right]$$

$R_2 = (-1)r_1 + r_2$:
$R_3 = (-1)r_1 + r_3$: $$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & c-1 & c-1 & -\frac{1}{c} & 1 & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$$

$R_2 = \left(\frac{1}{c-1}\right) r_2$: $$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{c} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$$

$R_1 = -r_2 + r_1$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c} + \frac{1}{c(c-1)} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 1 & -\frac{1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & c-1 & -\frac{1}{c} & 0 & 1 \end{array}\right]$$

$R_3 = \frac{1}{c-1} r_3$: $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 1 & \frac{-1}{c(c-1)} & \frac{1}{c-1} & 0 \\ 0 & 0 & 1 & \frac{-1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

$R_2 = (-1)r_3 + r_2$: $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{c-1} & \frac{-1}{c-1} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{c-1} & -\frac{1}{c-1} \\ 0 & 0 & 1 & -\frac{1}{c(c-1)} & 0 & \frac{1}{c-1} \end{array}\right]$$

$$\boxed{\therefore\ c \neq 0,\ c \neq 1.}$$

(b)[3] Find two non-zero matrices **A** and **B** such that $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.