Math 141 - Matrix Algebra for Engineers

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Matrices and Matrix Operations

Matrices

Matrices arise in other areas of mathematics. Let's study properties of matrices as mathematical objects in their own right

Definition: A rectangular array of numbers consisting of m horizontal rows and n vertical columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an $m \times n$ matrix or matrix of size $m \times n$. For entry a_{ij}, i is the row subscript, while j is the column subscript.

A general matrix is sometimes denoted $\begin{bmatrix} a_{ij} \end{bmatrix}_{m \times n}$.

Matrix Examples

Let

$$B = \left[\begin{array}{ccc} 1 & 6/7 & 4 \\ 1/2 & -7 & 3 \end{array} \right] .$$

The size of B is 2×3. A couple of entries of B are

$$b_{23} = 3, \qquad b_{21} = 1/2 .$$

Row and Column Vectors

- Let

$$\mathbf{p} = \begin{bmatrix} 3 & -4 & \pi \end{bmatrix}.$$

  p is called a *row matrix* or *row vector*. Here $p_1 = 3$, $p_2 = -4$ and $p_3 = \pi$.

- Let

$$\mathbf{q} = \begin{bmatrix} 0 \\ 2 \\ e \end{bmatrix}.$$

  q is called a *column matrix* or *column vector*. Here $q_1 = 0$, $q_2 = 2$ and $q_3 = e$.

Example

Example: Construct $\left[a_{ij}\right]_{4\times 3}$ if $a_{ij} = \dfrac{1}{i+j}$.

Solution: Let $A = \left[a_{ij}\right]_{4\times 3}$. The entries of A are functions of their row and column positions; that is, $a_{ij} = 1/(i+j)$ is a function of the two variables $i = 1, 2, 3, 4$ and $j = 1, 2, 3$:

$$a_{11} = \frac{1}{1+1} = \frac{1}{2}\, ,\ \ a_{12} = \frac{1}{1+2} = \frac{1}{3}\, ,\ \ a_{13} = \frac{1}{1+3} = \frac{1}{4}\, ,$$

and so on. Computing all of the entries in this way we have

$$A = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \frac{1}{1+3} \\ \frac{1}{2+1} & \frac{1}{2+2} & \frac{1}{2+3} \\ \frac{1}{3+1} & \frac{1}{3+2} & \frac{1}{3+3} \\ \frac{1}{4+1} & \frac{1}{4+2} & \frac{1}{4+3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}\, .$$

Square Matrices

A matrix with the same number *n* of rows and columns is called a **square matrix** of order *n*.

For example,

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

Here the entries $a_{11} = 1$, $a_{22} = 3$, and $a_{33} = -3$ (reading from upper-left to lower-right) form the **main diagonal** of *A*.

Equality of Matrices

The matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if A and B have the same size and $a_{ij} = b_{ij}$ for each i and j.

Example:

$$\left[\begin{array}{cc} 2 & 3 \\ 8 & 7 \end{array} \right] = \left[\begin{array}{cc} 1+1 & 3 \\ 16/2 & 9-2 \end{array} \right]$$

but

$$\left[\begin{array}{c} 2 \\ 2 \end{array} \right] \neq \left[\begin{array}{cc} 2 & 2 \end{array} \right]$$

since the sizes of are not the same: $\left[\begin{array}{c} 2 \\ 2 \end{array} \right]$ is size 2×1 while $\left[\begin{array}{cc} 2 & 2 \end{array} \right]$ is size 1×2.

Matrix Addition

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then

$$A + B = [a_{ij} + b_{ij}]_{m \times n} .$$

That is, provided A and B are the same size, the matrix $C = A + B$ is simply the $m \times n$ matrix formed by adding the corresponding entries of A and B: $c_{ij} = a_{ij} + b_{ij}$.

Example: Let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix} , \qquad B = \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix} .$$

Then

$$A + B = \begin{bmatrix} 3 + (-11) & 0 + 1 & (-2) + 12 \\ 4 + 7 & 3 + (-2) & 1 + 13 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 10 \\ 11 & 1 & 14 \end{bmatrix} .$$

Matrix Subtraction

If A and B are the same size, the matrix $C = A - B$ is the $m \times n$ matrix formed by subtracting the entries of B from the corresponding entries of A: $c_{ij} = a_{ij} - b_{ij}$.

Example: Let

$$A = \left[\begin{array}{ccc} 3 & 0 & -2 \\ 4 & 3 & 1 \end{array} \right], \qquad B = \left[\begin{array}{ccc} -11 & 1 & 12 \\ 7 & -2 & 13 \end{array} \right].$$

Then

$$A - B = \left[\begin{array}{ccc} 3 - (-11) & 0 - 1 & (-2) - 12 \\ 4 - 7 & 3 - (-2) & 1 - 13 \end{array} \right] = \left[\begin{array}{ccc} 14 & -1 & -14 \\ -3 & 5 & -12 \end{array} \right].$$

Scalar Multiplication

Let $A = [a_{ij}]_{m \times n}$ be a matrix and let k be a real number (a **scalar**). Then $C = kA$ is the matrix with entry $c_{ij} = ka_{ij}$.

For example, for

$$A = \left[\begin{array}{cc} 1 & 6 \\ 4 & 2 \end{array} \right],$$

multiplication by the scalar $1/2$ gives

$$\frac{1}{2}A = \frac{1}{2} \left[\begin{array}{cc} 1 & 6 \\ 4 & 2 \end{array} \right] = \left[\begin{array}{cc} (1/2)(1) & (1/2)(6) \\ (1/2)(4) & (1/2)(2) \end{array} \right] = \left[\begin{array}{cc} 1/2 & 3 \\ 2 & 1 \end{array} \right].$$

Matrix Multiplication

Definition: Let $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$. Then we define the product $C = AB$ as the matrix with entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj} .$$

Notice:

(i) $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{ip}$ are the elements of row i of A, while $b_{1j}, b_{2j}, b_{3j}, \ldots, b_{pj}$ are the elements of column j of B.

(ii) For matrix multiplication to be defined, the number of columns of A must be the same as the number of rows of B.

Matrix Multiplication Example

Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 6 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 7 \\ -2 & 5 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} (2)(3) + (3)(-2) & (2)(7) + (3)(5) \\ (1)(3) + (-2)(-2) & (1)(7) + (-2)(5) \\ (6)(3) + (4)(-2) & (6)(7) + (4)(5) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 29 \\ -1 & -3 \\ 10 & 62 \end{bmatrix}.$$

Notice A is size 3×2, B is size 2×2, and the product AB is size 2×2.

Matrix Multiplication: Another way to think about it

The process of multiplying matrices can be described in terms of a certain type of product. Let **R** and **C** be the row and column matrices

$$\mathbf{r} = \left[\begin{array}{cccc} r_1 & r_2 & \cdots & r_p \end{array} \right] \text{ and } \mathbf{c} = \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_p \end{array} \right] .$$

The expression $r_1 c_1 + r_2 c_2 + \cdots + r_p c_p$ is called the **dot product** of **r** and **c**. Using this notion, the product of general matrices A and B can be described as the matrix $D = AB$ which has entries

$d_{ij} =$ dot product of row i of A and column j of B .

Transpose of a Matrix

Suppose A is an $m \times n$ matrix. The **transpose** of A, denoted A^T, is the matrix of size $n \times m$ obtained by interchanging the rows and columns of A. For example, if

$$A = \left[\begin{array}{rrr} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{array} \right]$$

then

$$A^T = \left[\begin{array}{rrr} 1 & -2 & 5 \\ -2 & 3 & -7 \\ 1 & 1 & -3 \end{array} \right] .$$

Matrix Transpose Example

Let

$$B = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] \quad \Big\} \text{ size } 2 \times 3$$

then

$$B^\mathsf{T} = \left[\begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array} \right] \quad \Big\} \text{ size } 3 \times 2$$

Using this last example, notice that

$$(B^\mathsf{T})^\mathsf{T} = \left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right] = B \,.$$

This turns out to be true in general: for any matrix A, $(A^\mathsf{T})^\mathsf{T} = A$.

Trace of a Matrix

If A is a square matrix, the **trace** of A, written tr(A) is the sum of the entries on the main diagonal of A.

Example: For

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix},$$

$$\text{tr}(A) = 1 + 3 + (-3) = 1$$

Systems of Linear Equations as Matrix Products

Matrix multiplication can be used to express systems of linear equations.

The system

$$5x - 2y + z = -2$$
$$-x + 11y - 13z = 6$$
$$x + y + z = 1$$

is equivalent to

$$\begin{bmatrix} 5 & -2 & 1 \\ -1 & 11 & -13 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

Systems of Linear Equations as Matrix Products

That is . . .

$$\begin{bmatrix} 5 & -2 & 1 \\ -1 & 11 & -13 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

is equivalent to

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix of the system, and

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$