

Elementary Row Operations

The following **Elementary Row Operations (EROs)** change the form of a matrix without changing the solution of the corresponding system of equations:

1. Interchange any two rows.
2. Multiply a row through by a non-zero constant.
3. Add a constant of one row to another row.

Row Echelon Form

A matrix is said to be in **Row Echelon Form (REF)** if

1. The first non-zero entry in any row is a 1 (called a **leading 1**).
2. The leading 1 in any row is located to the right of the leading 1 of any row above.
3. Any rows consisting entirely of zeros are at the bottom of the matrix.

Example:

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is in REF, but

$$\begin{bmatrix} 1 & -4 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is not (why?).

Reduced Row Echelon Form

A matrix is said to be in **Reduced Row Echelon Form (RREF)** if, in addition to being in REF,

- Any column containing a leading 1 has zeros elsewhere in the column.

Example:

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in RREF, but

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is not (why?). Nor is

$$\begin{bmatrix} 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Gaussian Elimination Algorithm

To put a matrix into REF:

- Interchange rows (if necessary) so that the first non-zero entry in the top row is located as far to the left as possible.
- Use EROs to reduce the first entry in the top row to a leading 1. This can always be done by multiplying the top row by the reciprocal of the first entry in that row. A constant multiple of some other row can also be added to the top row to achieve this. For simplicity, avoid introducing fractions if possible.
- Now add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zero.
- Now, **without touching the top row**, go to step 1 and apply the procedure to the submatrix consisting of all rows below that containing the most recently used leading 1.
- Proceed until there are no more rows.

Gauss-Jordan Elimination Algorithm

To put a matrix into RREF, first put it into REF using Gaussian elimination procedure above, then perform an extra step:

6. Beginning with the last non-zero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.