Question 1: Solve the following system of equations **using either Gaussian or Gauss-Jordan elimination** (no credit will be given for using any other method). Use proper notation to state the row operations used at each step and clearly state the final solution.

$$\begin{array}{rcrcrcrcr} & & y & + & 2z & + & 2w & = & -6 \\ x & + & 2y & + & 5z & + & 3w & = & 2 \\ 2x & + & y & + & 5z & + & w & = & -3 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 2 & 2 & -6 \\ 1 & 2 & 5 & 3 & 2 \\ 2 & 1 & 5 & 1 & -3 \end{bmatrix}$$

$r_1 \leftrightarrow r_2$:

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 2 \\ 0 & 1 & 2 & 2 & -6 \\ 2 & 1 & 5 & 1 & -3 \end{bmatrix}$$

$R_3 = (-2)r_1 + r_3$:

$$\begin{bmatrix} 1 & 2 & 5 & 3 & 2 \\ 0 & 1 & 2 & 2 & -6 \\ 0 & -3 & -5 & -5 & -7 \end{bmatrix}$$

$R_1 = (-2)r_2 + r_1$:
$R_3 = 3r_2 + r_3$;

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 14 \\ 0 & 1 & 2 & 2 & -6 \\ 0 & 0 & 1 & 1 & -25 \end{bmatrix}$$

$R_1 = (-1)r_3 + r_1$:
$R_2 = (-2)r_3 + r_2$:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 39 \\ 0 & 1 & 0 & 0 & 44 \\ 0 & 0 & 1 & 1 & -25 \end{bmatrix}$$

$\therefore$ $w = t$

$z = -25 - w = -25 - t$

$y = 44$

$x = 39 + 2w = 39 + 2t$

$\therefore$ $(39+2t, 44, -25-t, t)$, $t \in \mathbb{R}$.

[10]

Question 2: Each of the following matrices in reduced row echelon form (RREF) was obtained from a system of linear equations. For each of the reduced matrices,

(a) If the solution is unique, write the solution.

(b) If there is more than one solution, properly express the solution using parameters.

(c) If there is no solution, then state "no solution"

(i) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (the variables here are x, y, z.)

no solution.

[1]

(ii) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (the variables here are x, y, z.)

$z=0, \quad y=t, \quad x=-2t$

$\therefore \; (-2t, t, 0), \quad t \in \mathbb{R}.$

[1]

(iii) $\begin{bmatrix} 1 & 0 & 4 & 5 & 1 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (the variables here are x, y, z, w.)

$$\left.\begin{aligned} z&=r, \; w=t \\ y&=2-r-3t \\ x&=1-4r-5t \end{aligned}\right\} \quad (1-4r-5t, 2-r-3t, r, t), \quad r,t \in \mathbb{R}.$$

[1]

(iv) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (the variables here are x, y, z.)

$x=5, \quad y=-1, \quad z=2.$

[1]

Question 3: If the system

$$\begin{aligned} 2x - 4y &= 1 \\ ax + 3y &= 2 \end{aligned}$$

has no solution what must be the value of a?

$$\begin{vmatrix} 2 & -4 \\ a & 3 \end{vmatrix} = 0 \Rightarrow 6+4a=0 \Rightarrow a = -\frac{6}{4} = \boxed{-\frac{3}{2}}$$

[2]

Question 4: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(a) Compute $\mathbf{AB}^{\mathsf{T}} - 2\mathbf{C}$ if it is defined.

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & -1 \\ 0 & 4 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 8 \\ -3 & 11 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 \\ -1 & 6 \\ -5 & 9 \end{bmatrix}}$$

[3]

(b) Compute $\operatorname{tr}((\mathbf{A} + \mathbf{I})(\mathbf{A} - \mathbf{I}))$ if it is defined.

$$(A+I)(A-I) = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 6 & 4 \\ 3 & 6 & 6 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 3 & 4 & 4 \\ 3 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 6 & \sim & \\ & 51 & \\ \sim & & 51 \end{bmatrix}$$

$$\therefore \operatorname{tr}((A+I)(A-I)) = 6+51+51 = \boxed{108}$$

[2]

(c) Determine $\mathbf{A}^{-1}$ if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 4 & 0 & 1 & 0 \\ 3 & 6 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = (-3)r_1 + r_2$:
$R_3 = (-3)r_1 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 3 & 2 & -3 & 0 & 1 \end{array}\right]$$

$r_2 \leftrightarrow r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & -3 & 0 & 1 \\ 0 & 2 & 1 & -3 & 1 & 0 \end{array}\right]$$

$R_2 = (-1)r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & -3 & 1 & 0 \end{array}\right]$$

$R_1 = (-1)r_2 + r_1$:
$R_3 = (-2)r_2 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -3 & 3 & -2 \end{array}\right]$$

$R_3 = (-1)r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

$R_2 = (-1)r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

$$\boxed{\therefore A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}}$$

[5]

Question 5: Determine the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$.

(Row operations will make this much easier.)

$R_i = (-1)r_1 + r_i$, $i = 2,3,4,5$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\therefore \det(A) = (1)(1)(3)(2)(4) = \boxed{24}$$

[4]

Question 6: Use Cramer's rule to solve for y:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 4y + 5z &= -2 \\ x + y + 2z &= -1 \end{aligned}$$

Note that $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{vmatrix} = 6$. $= \det(A)$ say

$$|A_2| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = (1)(-4+5) - (1)(4-5) + \cancel{(-1)(-2-(-2))}^{\,0} = 2$$

$$\therefore y = \frac{|A_2|}{|A|} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

[4]

Question 7: Suppose **u**, **v** and **w** are any vectors in R^3. Circle true (T) or false (F), as appropriate, for the following statements:

(i) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ T F

(ii) $\mathbf{u} \times (\mathbf{u} \cdot \mathbf{v}) = 0$ T F

(iii) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ T F

(iv) $\text{proj}_{\mathbf{u}}\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u}$ T F

(v) If **u**, **v** and **w** are linearly independent, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ T F

[5]

Question 8: Determine the equation of the line in R^3 that contains the point $(-1, 6, 0)$ and is orthogonal to the plane $4x + 2y - z = 5$. You may state your answer in either vector or parametric form.

$\vec{n} = (4, 2, -1)$ = direction vector of line

$$\therefore \vec{r}(t) = (-1, 6, 0) + t(4, 2, -1)$$

$$= \boxed{(-1 + 4t, 6 + 2t, -t)}$$

[3]

Question 9: Find an equation of the plane passing through the points $P_1(-1, 0, 4)$, $P_2(-1, 4, 3)$ and $P_3(0, 6, -2)$. State your answer in the form

$$ax + by + cz = d$$

for appropriate constants a, b, c and d. (We saw three different methods for doing this problem; use whichever method you like.)

$\vec{u} = \overrightarrow{P_1P_2} = (0, 4, -1)$

$\vec{v} = \overrightarrow{P_1P_3} = (1, 6, -6)$

normal to plane is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 4 & -1 \\ 1 & 6 & -6 \end{vmatrix}$$

$$= -18\hat{\imath} - \hat{\jmath} - 4\hat{k}$$

$$= (-18, -1, -4)$$

using $P_1(-1, 0, 4)$:

$$(-18, -1, -4) \cdot (x+1, y-0, z-4) = 0$$

$$-18x - 18 - y - 4z + 16 = 0$$

$$\boxed{-18x - y - 4z = 2}$$

[5]

Question 10: Suppose $\mathbf{v}_1 = (1, a, 2)$, $\mathbf{v}_2 = (0, 2, a)$ and $\mathbf{v}_3 = (1, 1, 1)$ are linearly dependent. What are the possible values for a?

$$\begin{vmatrix} 1 & 0 & 1 \\ a & 2 & 1 \\ 2 & a & 1 \end{vmatrix} = 0 \Rightarrow (1)(2-a) - (0)(\sim) + (1)(a^2-4) = 0$$

$$\Rightarrow a^2 - a - 2 = 0$$

$$(a-2)(a+1) = 0$$

$$\boxed{a=2,\ a=-1}$$

[4]

Question 11: Let $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$ and $\mathbf{w} = (2, -3)$. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$. That is, find $(\mathbf{w})_S$.

$$a\vec{u}_1 + b\vec{u}_2 = \vec{w}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \end{bmatrix}$$

$R_2 = r_1 + r_2$: $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$

$R_2 = +\frac{1}{2} r_2$: $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$

$R_1 = (-1) r_2 + r_1$: $\begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}$

$\therefore (a, b) = \left(\frac{5}{2}, \frac{-1}{2}\right)$

$$\boxed{\therefore (\vec{w})_S = \left(\frac{5}{2}, \frac{-1}{2}\right)}$$

[4]

Question 12: Let $\mathbf{p}_1 = 1 + 2x + x^2$ and $\mathbf{p}_2 = 2 - x^2$. Find a third vector $\mathbf{p}_3$ so that $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2 . (There are many possible correct answers here; give one answer with justification.)

$\dim(P_2) = 3$, so enough to find $\vec{p}_3 = a + bx + cx^2$ so that

$$\begin{vmatrix} 1 & 2 & a \\ 2 & 0 & b \\ 1 & -1 & c \end{vmatrix} \neq 0 \Rightarrow (1)(0+b) - 2(2c-b) + a(-2-0) \neq 0$$

$$\Rightarrow -2a + 3b - 4c \neq 0$$

so take $a=1$, $b=c=0$: $\boxed{\vec{p}_3 = 1}$

is one solution.

[4]

Question 13: The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 1 & 2 & 1 & 0 & 2 \\ 3 & 6 & 1 & 0 & 8 \\ -1 & -2 & -2 & 0 & -1 \end{bmatrix}$ has RREF $\mathbf{R} = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \end{array}$.

(a) Give a basis and state the dimension of the row space of **A** .

basis $S_{row} = \{(1,2,0,0,3), (0,0,1,0,-1)\}$

dimension = 2

[2]

(b) Give a basis and state the dimension of the column space of **A** .

basis $S_{col} = \{(1,1,3,-1), (0,1,1,-2)\}$

dimension = 2,

[2]

(c) Give a basis and state the dimension of the null space of **A** . (Equivalently, give a basis and state the dimension of the solution space of $\mathbf{Ax} = \mathbf{0}$.)

$x_2 = r,\ x_4 = s,\ x_5 = t,\ x_3 = t,\ x_1 = -3t - 2r.$

$\therefore$ solutions are $(-2r-3t, r, t, s, t)$

$= r(-2,1,0,0,0) + s(0,0,0,1,0) + t(-3,0,1,0,1)$

$\therefore$ basis is $\{(-2,1,0,0,0), (0,0,0,1,0), (-3,0,1,0,1)\}$, dim = 3. [3]

(d) Determine rank(**A**) and nullity(**A**) .

rank(A) = 2 , nullity(A) = 3

[1]

(e) Determine rank($\mathbf{A}^T$) and nullity($\mathbf{A}^T$) .

rank(A^T) = 2 , nullity(A^T) = 2

[1]

Question 14:

Let the transformation T_1 represent reflection about the yz-plane:

$$T_1\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} -a \\ b \\ c \end{bmatrix},$$

and T_2 be the transformation which rotates a vector in R^3 counter-clockwise about the z-axis by an angle θ.

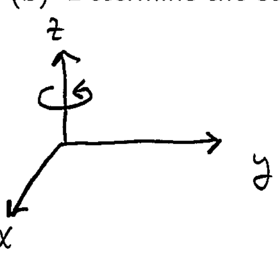
(a) Determine the standard matrix **A** for the transformation T_1.

$$T_1\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad T_1\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

[2]

(b) Determine the standard matrix **B** for the transformation T_2.



$$T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad T_2\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

[3]

(c) Determine the image of the vector $(1, 2, -1)$ if it is first reflected about the yz-plane and then rotated about the z-axis by $\pi/4$ (or 45°.)

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \\ -1 \end{bmatrix}$$

[3]

(d) For any vector (a, b, c), will applying the transformation T_1 followed by T_2 produce the same result as first applying T_2 followed by T_1? Explain using matrices.

$T_2 \circ T_1$ has standard matrix $BA = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_1 \circ T_2$ has standard matrix $AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $BA \neq AB$ for, say, $\theta = \frac{\pi}{4}$, the result $\boxed{\text{will not be the same.}}$

[2]

Question 15: Find the eigenvalues and corresponding eigenvectors (that is, the eigenpairs) for $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$.

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow 6 - 5\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-4) = 0$$

$$\Rightarrow \lambda = 1, \ \lambda = 4.$$

$\underline{\lambda = 1:}$

$$A\vec{u} = 1\vec{u} \Rightarrow (A - I)\vec{u} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

$$\therefore \lambda_1 = 1, \ \vec{u}_1 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{u} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\underline{\lambda = 4:}$

$$A\vec{u} = 4\vec{u} \Rightarrow (A - 4I)\vec{u} = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} = 0$$

$$\therefore \lambda_2 = 4, \ \vec{u}_2 = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \vec{u} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = t'\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

[6].