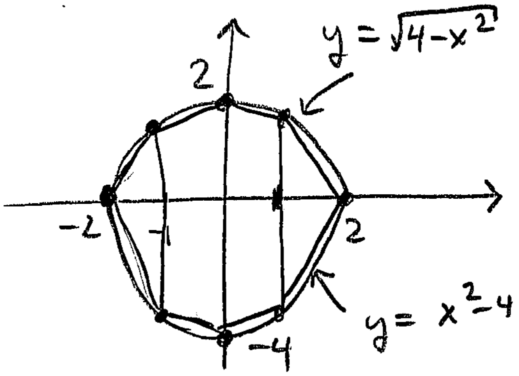
Question 1 [10 points]:

(a) Set up the integral representing the area bounded between the curves $y = \sqrt{4-x^2}$ (the top half of a circle) and $y = x^2 - 4$ over $-2 \le x \le 2$.



$$A = \int_{-2}^{2} \sqrt{4-x^2} - (x^2-4)\, dx$$

[3]

(b) Use T_4, the trapezoid rule on four subintervals, to approximate the integral in part (a).

$$\Delta x = \frac{2-(-2)}{4} = 1, \quad f(x) = \sqrt{4-x^2} - (x^2-4)$$

$$T_4 = \frac{\Delta x}{2}\left[f(-2) + 2f(-1) + 2f(0) + 2f(1) + f(2) \right]$$

$$= \frac{1}{2}\left[0 + 2(\sqrt{3}+3) + 2(6) + 2(\sqrt{3}+3) + 0 \right]$$

$$= 12 + 2\sqrt{3}.$$

[5]

(c) Is your approximation in part (b) an overestimate or underestimate of the exact area? Explain.

Underestimate: approximating trapezoids lie entirely within but do not cover the region of interest.

[2]

Question 2 [10 points]:

(a) Evaluate the improper integral or show that is it divergent. Clearly and neatly show all details, including any required substitutions or limits.

$$\int_0^1 \frac{e^x}{\sqrt{e^x - 1}}\,dx$$

$$= \lim_{a \to 0^+} \int_a^1 \frac{e^x}{\sqrt{e^x-1}}\,dx$$

let $u = e^x - 1$
$du = e^x\,dx$
$x = a \Rightarrow u = e^a - 1$
$x = 1 \Rightarrow u = e - 1$

$$= \lim_{a \to 0^+} \int_{e^a-1}^{e-1} u^{-\frac{1}{2}}\,du$$

$$= \lim_{a \to 0^+} \left[2u^{1/2}\right]_{e^a-1}^{e-1}$$

$$= \lim_{a \to 0^+} \left(2\sqrt{e-1} - 2\sqrt{e^a-1}\right)$$

$$= \boxed{2\sqrt{e-1}}$$

[5]

(b) Use the comparison theorem to determine whether $\int_1^\infty \frac{x}{e^x + x^4}\,dx$ converges or diverges. Be sure to state any theorems or results used.

On $[1,\infty)$, $0 \leq \frac{x}{e^x+x^4} \leq \frac{x}{0+x^4} = \frac{1}{x^3}$.

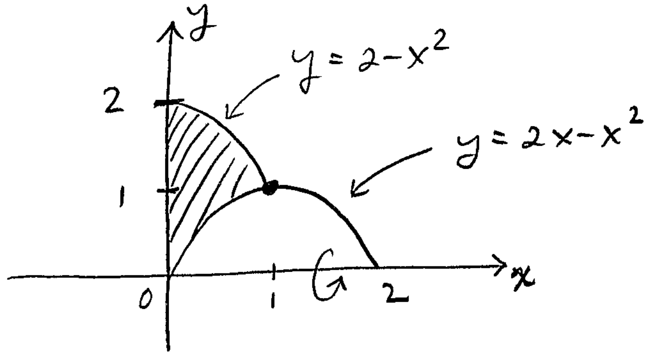
Since $\int_1^\infty \frac{1}{x^3}\,dx$ converges (p-integral, $p>1$),

so does $\int_1^\infty \frac{x}{e^x+x^4}\,dx$ by the comparison thm.

[5]

Question 3 [10 points]:

(a) The region in the first quadrant that is bounded by $y = 2x - x^2$, $y = 2 - x^2$ and the y-axis is rotated about the x-axis. Determine the volume of the resulting solid. (The washer method would be best here.)



$$V = \int_0^1 \pi(2-x^2)^2 - \pi(2x-x^2)^2\,dx$$

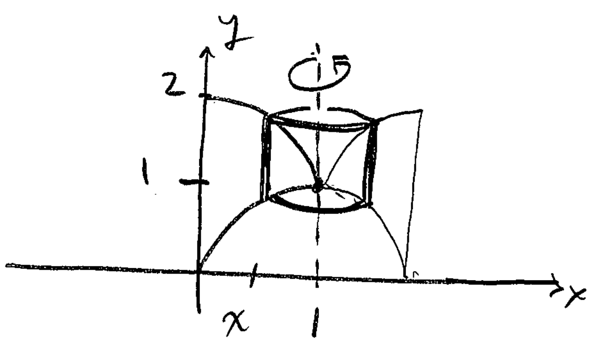
$$= \pi\int_0^1 4 - 4x^2 \cancel{+x^4} - 4x^2 + 4x^3 \cancel{-x^4}\,dx$$

$$= 4\pi\left[x - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = 4\pi\left(1 - \frac{2}{3} + \frac{1}{4}\right)$$

$$= 4\pi\left(\frac{12-8+3}{12}\right) = \boxed{\frac{7\pi}{3}}$$

[5]

(b) The same region as in part (a) is rotated about the vertical line $x = 1$. Determine the volume of the resulting solid. (Cylindrical shells would be best here.)



$$V = \int_0^1 2\pi(1-x)\left((2-x^2) - (2x-x^2)\right)dx$$

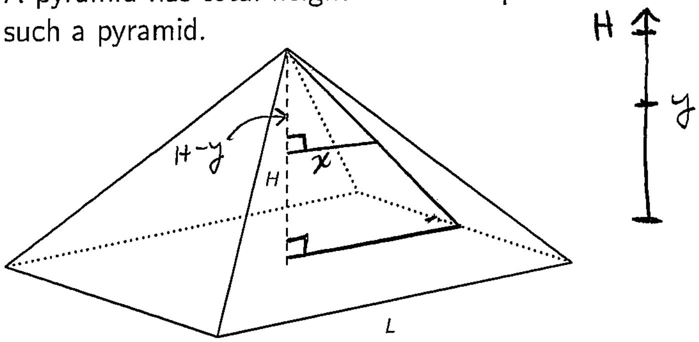
$$= 2\pi\int_0^1 (1-x)(2-2x)\,dx$$

$$= 4\pi\int_0^1 (1-x)^2\,dx$$

$$= \frac{4\pi}{-3}\left[(1-x)^3\right]_0^1 = \boxed{\frac{4\pi}{3}}$$

[5]

Question 4 [5 points]:
A pyramid has total height H and a square base of side length L. Use integration to determine the volume of such a pyramid.



By similar triangles: $\frac{H-y}{x} = \frac{H}{(L/2)} \Rightarrow x = \frac{L}{2H}(H-y)$.

$\therefore$ horizontal slice at y has cross-sectional area

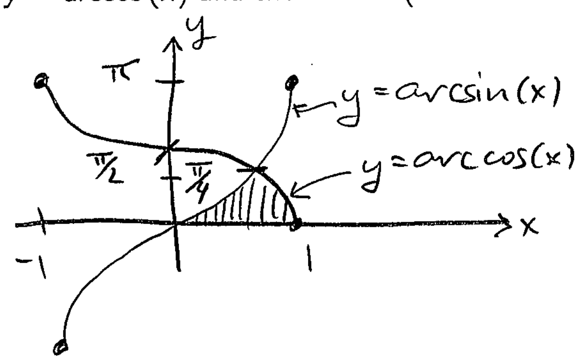
$$A(y) = (2x)^2 = 4x^2 = 4\frac{L^2}{4H^2}(H-y)^2 = \frac{L^2}{H^2}(H-y)^2$$

$\therefore$ volume is

$$V = \int_0^H \frac{L^2}{H^2}(H-y)^2\,dy$$
$$= \frac{L^2}{H^2}\left.\frac{(H-y)^3}{-3}\right|_0^H$$
$$= \frac{L^2}{H^2}\frac{H^3}{3} = \boxed{\frac{1}{3}L^2 H}$$

[5]

Question 5 [5 points]: Determine the area of the region in the first quadrant that is bounded by $y = \arcsin(x)$, $y = \arccos(x)$ and the x axis. (Hint: think about x as a function of y.)



$y = \arcsin(x) \Rightarrow x = \sin y$

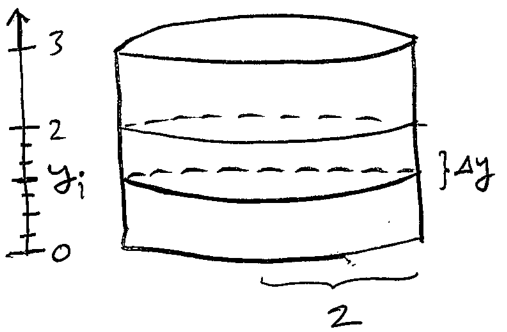
$y = \arccos(x) \Rightarrow x = \cos y$.

$$\therefore A = \int_{y=0}^{\pi/4} \cos y - \sin y \, dy$$
$$= \left[\sin(y) + \cos(y)\right]_0^{\pi/4}$$
$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \boxed{\sqrt{2} - 1}$$

[5]

Question 6 [10 points]:

(a) A cylindrical vessel of top radius 2 m and height 3 m is filled with water to a depth of 2 m. Determine the amount of work required to empty the vessel by pumping water up and over the top rim. Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2. You may leave the constants ρ and g in your final answer if you like. State units with your answer.



slice of water at y_i has weight

$$F_{y_i} = (\text{volume})(\text{density})\, g = (\pi r^2 h)(\rho)(g) = 4\pi \Delta y \rho g \quad N$$

Work to lift slice at y to top of vessel is

$$W_{y_i} = (F_{y_i})(\text{distance}) = (4\pi\rho g \Delta y)(3-y_i) \quad J$$

$\therefore$ total work to empty vessel is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_{y_i} = \int_0^2 4\pi\rho g(3-y)\,dy$$

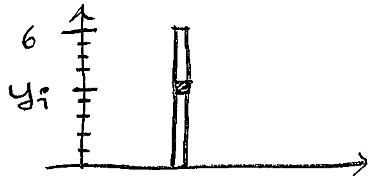
$$= 4\pi\rho g\left[3y - \frac{y^2}{2}\right]_0^2$$

$$= 4\pi\rho g \cdot (4-0) = \boxed{16\pi\rho g \ J}$$

[5]

(b) A chain lying on the ground is 10 m long and has a mass of 80 kg. Use integration to determine the amount of work required to raise one end of the chain to a height of 6 m . Again, recall that acceleration due to gravity is $g = 9.8$ m/s^2 (you may leave the constant g in your final answer if you like.)

Chain has linear density $\frac{80 \text{ kg}}{10 \text{ m}} = 8 \frac{\text{kg}}{\text{m}}$.



Segment at y_i has weight

$$F_{y_i} = (\text{length})(\text{linear density})\, g = 8g\,\Delta y$$

Work to lift segment from ground to height y_i is

$$W_{y_i} = (F_{y_i})(\text{distance}) = 8g\, y_i\, \Delta y$$

$\therefore$ Total work to lift all such segments is then

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_{y_i} = \int_0^6 8gy\,dy = 8g\left.\frac{y^2}{2}\right|_0^6 = \boxed{144g \ J}$$

[5]