Question 1 [10 points]:

(a) John and Mary are each setting up retirement plans. John will deposit \$200 at the end of each month for the next 30 years into a fund which pays 6% compounded monthly. Mary found a different investment which offers an interest rate of 7% compounded annually, and so she decides to make equal annual contributions to this second fund. Her goal is to match John's fund at the end of the 30 years. How much should she deposit each year? (Round your final answer to two decimal places.)

John: $P = 200$, $i = \frac{0.06}{12} = 0.005$, $m = (12)(30) = 360$

$$\therefore A = P\left[\frac{(1+i)^m - 1}{i}\right] = 200\left[\frac{(1.005)^{360} - 1}{0.005}\right] \doteq \$200{,}903.01$$

Mary: $A = 200{,}903.01$, $i = 0.07$, $m = 30$.

$$\therefore P = \frac{iA}{(1+i)^m - 1} = \frac{(0.07)(200{,}903.01)}{(1.07)^{30} - 1} \doteq \boxed{\$2126.84}$$

[5]

(b) Smithers has just won the lottery. He has the option of taking the \$10,000,000 prize as a single lump sum now, or he can elect to receive payments of \$8000 at the end of each week for the next 25 years. If Smithers is confident that he can invest the \$10,000,000 at 5% compounded weekly, would he be better off taking the weekly payment option or taking the lump sum and investing it?

Let V = present value of the \$8000 annuity.

$P = 8000$, $i = \frac{0.05}{52}$, $m = (52)(25) = 1300$.

$$V = P\left[\frac{1 - (1+i)^{-m}}{i}\right] = 8000\left[\frac{1 - \left(1 + \frac{0.05}{52}\right)^{-1300}}{\left(\frac{0.05}{52}\right)}\right] \doteq \$5{,}934{,}848.$$

Since $V < 10{,}000{,}000$, he should take the lump-sum payment.

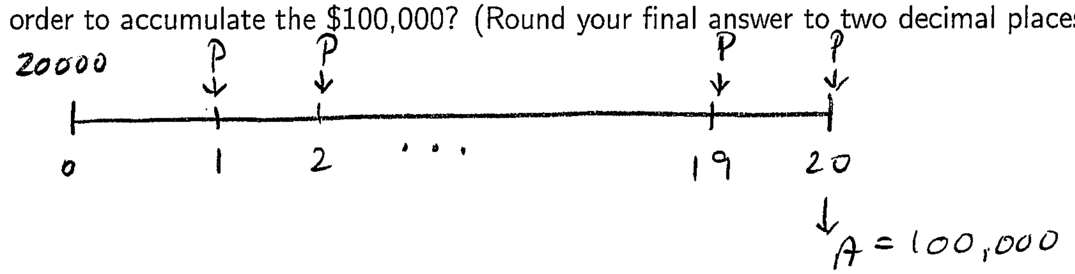
or: Investing $V = \$10{,}000{,}000$ yields weekly payments of

$$P = \frac{\left(\frac{0.05}{52}\right)(10{,}000{,}000)}{\left[1 - (1 + 0.05/52)^{-1300}\right]} = \$13{,}479.70 > \$8000.$$

[5]

Question 2 [10 points]:

(a) A condominium building will require a new roof in 20 years time at a projected cost of \$100,000. The condominium committee currently has \$20,000 in their building maintenance account and they will accumulate the rest of the funds necessary by making equal deposits at the end of each year for the next 20 years. The account earns interest at 4.5% compounded annually. What should the annual deposits be in order to accumulate the \$100,000? (Round your final answer to two decimal places.)



$i = 4.5\%$, $m = 20$ for the annuity

Let A_1 = future value of the initial 20,000.

$$= 20000(1+0.045)^{20}$$

Let A_2 = amount of the annuity $= P\left[\frac{(1+0.045)^{20}-1}{0.045}\right]$.

We need $A_1 + A_2 = 100\,000$

$$\therefore\ 20\,000(1.045)^{20} + P\left[\frac{(1.045)^{20}-1}{0.045}\right] = 100\,000$$

$$\therefore\ P = \frac{100000 - 20\,000(1.045)^{20}}{\left[\frac{(1.045)^{20}-1}{0.045}\right]} \doteq \boxed{\$1650.09}$$

[5]

(b) A \$300,000 bank loan will be repaid by making payments at the end of every month. One bank offers 4% interest compounded monthly with a 20 year repayment period, while another bank offers 5% compounded monthly with a 25 year repayment period. Which repayment option results in the lowest monthly payment?

Bank 1: $V = 300,000$, $i = \frac{0.04}{12}$, $m = (20)(12) = 240$.

$$\therefore\ P = \frac{iV}{1-(1+i)^{-m}} = \frac{\left(\frac{0.04}{12}\right)(300,000)}{1-\left(1+\frac{0.04}{12}\right)^{-240}} \doteq \$1817.94$$

Bank 2: $V = 300000$, $i = \frac{0.05}{12}$, $m = (25)(12) = 300$.

$$\therefore\ P = \frac{iV}{1-(1+i)^{-m}} = \frac{\left(\frac{0.05}{12}\right)(300000)}{1-\left(1+\frac{0.05}{12}\right)^{-300}} = \boxed{\$1753.77}$$

$\therefore$ The second repayment option results in a lower monthly payment.

[5]

Question 3 [10 points]:

(a) For this question use the following sets:

$$U = \{a, b, c, d, e, f\}, \quad A = \{b, c\}, \quad B = \{c, d, e\}, \quad C = \{a, f\}$$

Determine the following:

(i) $\overline{A} \cap B$ $= \{a, d, e, f\} \cap \{c, d, e\}$

$= \boxed{\{d, e\}}$

[2]

(ii) $(B \cap C) \cup (B \cap \overline{C})$ $B \cap C = \{c, d, e\} \cap \{a, f\} = \emptyset$

$B \cap \overline{C} = \{c, d, e\} \cap \{b, c, d, e\}$

$= \{c, d, e\} = B$

$\therefore (B \cap C) \cup (B \cap \overline{C}) = \emptyset \cup B = B = \boxed{\{c, d, e\}}$

[2]

(iii) $\overline{(\overline{A} \cap \overline{C})}$ $\overline{A} \cap \overline{C} = \{a, d, e, f\} \cap \{b, c, d, e\}$

$= \{d, e\}$

$\therefore \overline{(\overline{A} \cap \overline{C})} = \boxed{\{a, b, c, f\}}$

[3]

(b) Suppose A and B are sets with the property that $n(A \cup B) = n(A \cap B)$. What is $n(A) - n(B)$?

Notice: $n(A \cap B) \leq n(A) \leq n(A \cup B)$

and $n(A \cap B) \leq n(B) \leq n(A \cup B)$

If $n(A \cap B) = n(A \cup B)$, then $n(A) = n(B)$, so

$\boxed{n(A) - n(B) = 0}$

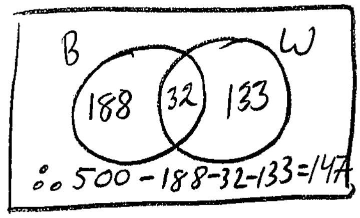
[3]

Question 4 [10 points]:

(a) A survey of 500 students found that during the last week 220 had eaten at Burger King, 165 had eaten at Wendy's, while 32 had eaten at both places. How many of the 500 had eaten at neither of the two places?

B : Burger King

W : Wendy's



∴ $\boxed{\text{147 ate at neither.}}$

[3]

(b) A telephone number has seven digits selected from $0, 1, 2, \ldots, 9$. Repetition of digits within a phone number is allowed, but the first digit cannot be 0 or 1. How many different telephone numbers are possible?

Choices for digit : 8 10 10 10 10 10 10

∴ number of different telephone numbers is $8 \cdot 10^6 = \boxed{8{,}000{,}000}$

[3]

(c) Again, a telephone number has seven digits selected from $0, 1, 2, \ldots, 9$. Repetition of digits within a phone number is allowed, but the first digit cannot be 0 or 1. How many different telephone numbers have at least one repeated digit?

$$\begin{bmatrix}\text{number with at} \\ \text{least one repeated} \\ \text{digit}\end{bmatrix} = \begin{bmatrix}\text{total number} \\ \text{possible}\end{bmatrix} - \begin{bmatrix}\text{number with} \\ \text{no repeated} \\ \text{digits}\end{bmatrix}$$

$$= 8\,000\,000 - (8)(9)(8)(7)(6)(5)(4)$$

$$= \boxed{7{,}516{,}160}$$

[4]

Question 5 [10 points]:

(a) 4-letter codes are made using the 26-letters of the alphabet and repetition of letters within a code is permitted. What is the probability that a code formed at random consists of the same letter repeated 4 times?

$S = \{(l_1, l_2, l_3, l_4) \mid l_1, l_2, l_3, l_4 \text{ are letters of alphabet}\}$.

$n(S) = (26)(26)(26)(26) = 26^4$.

$E = \{(l_1, l_2, l_3, l_4) \mid l_1, l_2, l_3, l_4 \text{ are letters of alphabet and } l_1 = l_2 = l_3 = l_4\}$

$n(E) = 26$.

Outcomes are equally likely, so

$P(E) = \frac{n(E)}{n(S)} = \frac{26}{26^4} = \frac{1}{26^3} \doteq \boxed{0.000\,057}$ **[3]**

(b) In a group of six people what is the probability that at least two have their birthdays in the same calendar month?

Let E = "at least two have birthdays in same month"

$\bar{E}$ = "no two have birthday in same month".

S = "all possible birth month assignments to 6 people"

$n(S) = 12^6$.

$n(\bar{E}) = (12)(11)(10)(9)(8)(7)$

$P(E) = 1 - P(\bar{E}) = 1 - \frac{n(\bar{E})}{n(S)} = 1 - \frac{(12)(11)(10)(9)(8)(7)}{12^6} \doteq \boxed{0.78}$

[3]

(c) A contestant rolls a single die and receives \$5 if the outcome is a ⚀ or ⚁, \$15 if the outcome is a ⚂, and \$2 if it is a ⚃, ⚄ or ⚅. Determine the expected payout of the game. (Round your final answer to two decimal places.)

$m_1 = 5$, E_1 = "roll a ⚀ or ⚁", $P(E_1) = \frac{2}{6}$.

$m_2 = 15$, E_2 = "roll a ⚂", $P(E_2) = \frac{1}{6}$.

$m_3 = 2$, E_3 = "roll a ⚃, ⚄ or ⚅", $P(E_3) = \frac{3}{6}$.

$E = m_1 P(E_1) + m_2 P(E_2) + m_3 P(E_3)$

$= (5)\left(\frac{2}{6}\right) + (15)\left(\frac{1}{6}\right) + (2)\left(\frac{3}{6}\right) = \frac{31}{6} \doteq \boxed{\$5.17}$ **[4]**