Question 1:

(a)[5] Use S_4, Simpson's rule on 4 subintervals, to approximate $\int_1^3 \frac{1}{x^2}\,dx$. (You may leave your final answer in a "calculator ready" form, that is, as a sum of fractions.)

$f(x) = \frac{1}{x^2}$, $\Delta x = \frac{3-1}{4} = \frac{1}{2}$

$$\therefore S_4 = \frac{\Delta x}{3}\left[f(1) + 4f(\tfrac{3}{2}) + 2f(2) + 4f(\tfrac{5}{2}) + f(3)\right]$$

$$= \frac{(\frac{1}{2})}{3}\left[\frac{1}{1^2} + 4\frac{1}{(\frac{3}{2})^2} + 2\frac{1}{2^2} + 4\frac{1}{(\frac{5}{2})^2} + \frac{1}{3^2}\right]$$

$$= \frac{1}{6}\left[1 + \frac{16}{9} + \frac{1}{2} + \frac{16}{25} + \frac{1}{9}\right] \checkmark$$

$$= \frac{1813}{2700}$$

(b)[5] Suppose you are instead using the trapezoid rule on 4 subintervals to approximate $\int_1^3 \frac{1}{x^2}\,dx$. Determine an error bound $|E_{T_4}|$ on the resulting approximation.

Recall: the error in using the trapezoid rule to approximate $\int_a^b f(x)\,dx$ using n subintervals is at most $\frac{K(b-a)^3}{12n^2}$ where $|f''(x)| \leq K$ on $[a, b]$.

$f(x) = x^{-2}$; $f'(x) = -2x^{-3}$; $f''(x) = 6x^{-4}$.

$\therefore$ on $[1,3]$ $|f''(x)| \leq |f''(1)| = 6 = K$.

$$\therefore |E_{T_4}| \leq \frac{K(b-a)^3}{12n^2} = \frac{6(3-1)^3}{12(4)^2} = \boxed{\frac{1}{4}}$$

Question 2:

(a)[5] Evaluate the improper integral $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ making proper use of all required limits.

For $I = \int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$, let $u = -\sqrt{x}$, $du = \frac{-1}{2\sqrt{x}}\,dx$

$$\therefore\ I = -2\int \frac{-e^{-\sqrt{x}}}{2\sqrt{x}}\,dx = -2\int e^u\,du = -2e^u = -2e^{-\sqrt{x}}.$$

$$\therefore\ \int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[-2e^{-\sqrt{x}}\right]_t^1$$

$$= \lim_{t\to 0^+}\left[-2e^{-1} - (-2e^{-\sqrt{t}})\right]$$

$$= \boxed{2\left(1-\frac{1}{e}\right)}$$

(b)[5] Evaluate the improper integral $\int_2^\infty \frac{1}{x\ln(x)}\,dx$ making proper use of all required limits.

For $I = \int \frac{1}{x\ln(x)}\,dx$, let $u = \ln(x)$, $du = \frac{1}{x}\,dx$.

$$\therefore\ I = \int \frac{1}{u}\,du = \ln(u) = \ln(\ln(x)).$$

$$\therefore\ \int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{t\to\infty}\left[\ln(\ln(x))\right]_2^t$$

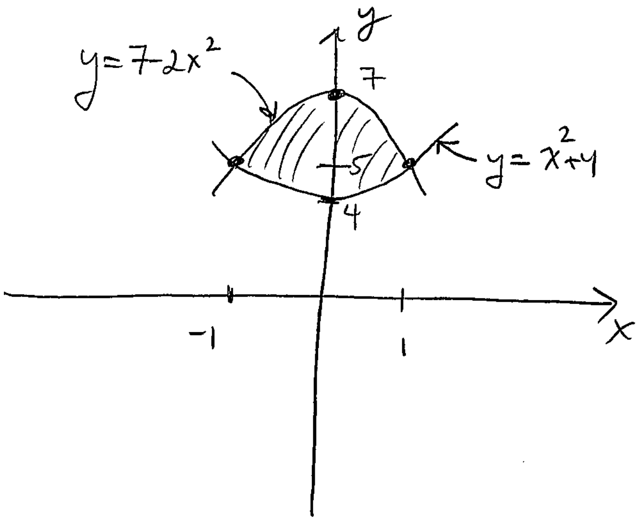
$$= \lim_{t\to\infty}\left[\ln(\ln(t)) - \ln(\ln(2))\right]$$

$$= \boxed{\infty}$$

Question 3:

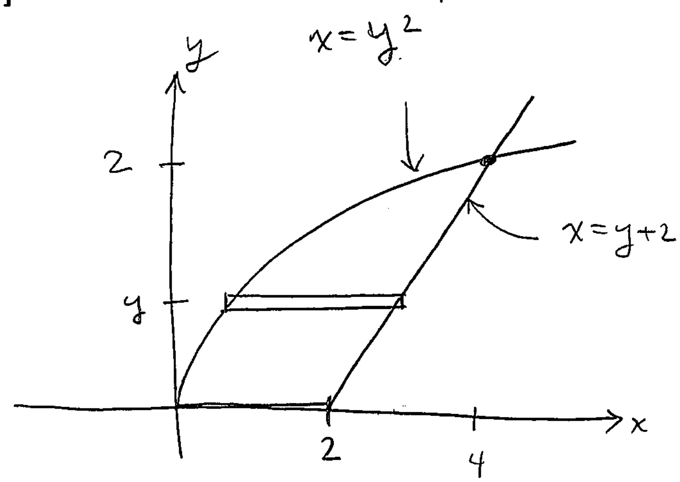
(a)[5] Determine the area of the region bounded between the curves $y = 7 - 2x^2$ and $y = x^2 + 4$.

$$7 - 2x^2 = x^2 + 4$$
$$3x^2 - 3 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$



$$\therefore A = \int_{-1}^{1} (7-2x^2) - (x^2+4)\, dx$$
$$= 2\int_0^1 3 - 3x^2\, dx$$
$$= 2\left[3x - x^3\right]_0^1$$
$$= 2\left[3 - 1\right]$$
$$= \boxed{4}$$

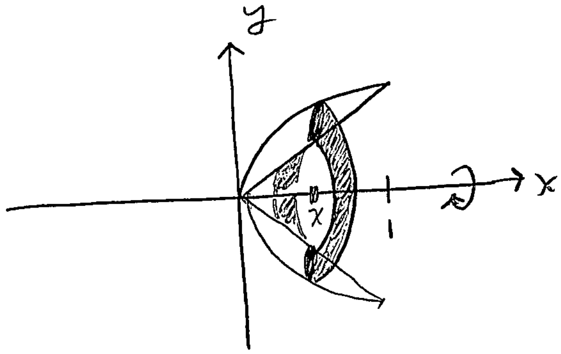
(b)[5] Determine the area in the first quadrant that is bounded by $x = y^2$, $x = y + 2$ and the x-axis.



$$A = \int_{y=0}^{2} (y+2) - y^2\, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2$$
$$= \frac{2^2}{2} + 2(2) - \frac{2^3}{3}$$
$$= 6 - \frac{8}{3}$$
$$= \boxed{\frac{10}{3}}$$

Question 4:

(a)[7] The region in the first quadrant that is bounded by $y = \sqrt{x}$ and $y = x$ is rotated about the x-axis. Determine the volume of the resulting solid.



$$V = \int_0^1 \pi(\sqrt{x})^2 - \pi(x)^2\,dx$$

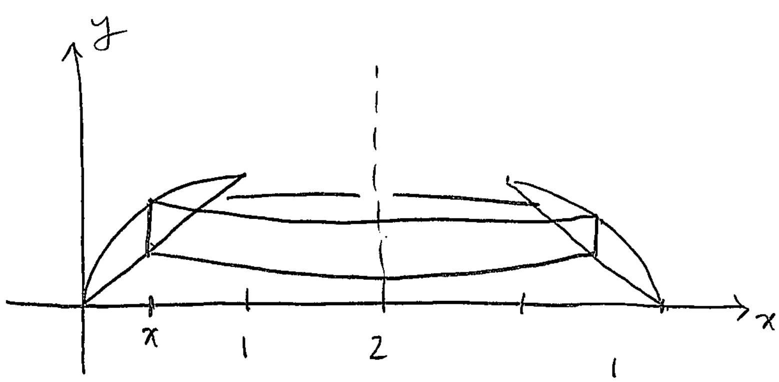
$$= \pi \int_0^1 x - x^2\,dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \pi\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \boxed{\frac{\pi}{6}}$$

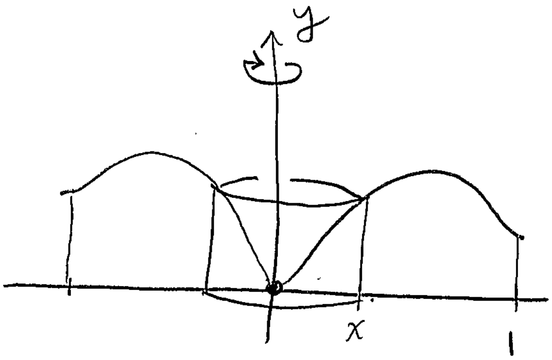
(b)[3] The same region as in part (a) is rotated about the vertical line $x = 2$. Set up BUT DO NOT EVALUATE an integral representing the volume of the resulting solid.



By cylindrical shells: $V = \int_{x=0}^{1} 2\pi(2-x)(\sqrt{x} - x)\,dx$

Question 5:

(a)[7] The region in the first quadrant that is bounded by $y = 3xe^{-x^3}$ and the lines $x = 0$ and $x = 1$ is rotated about the y-axis. Determine the volume of the resulting solid.



By cylindrical shells:

$$V = \int_0^1 2\pi x \cdot 3xe^{-x^3}\,dx$$

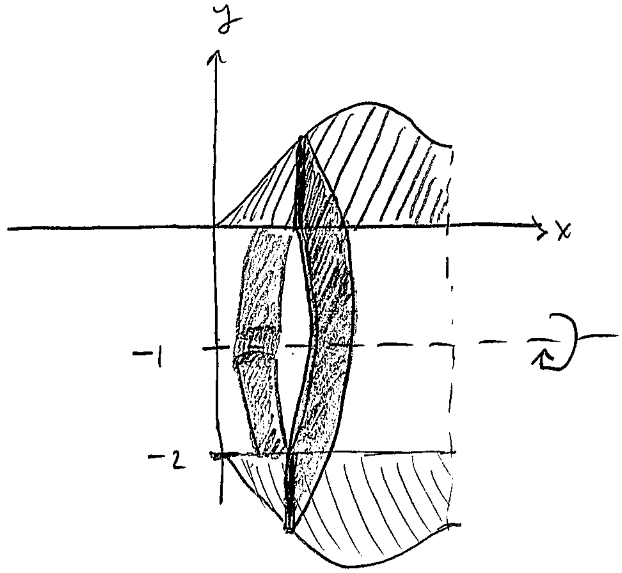
$$= -2\pi \int_0^1 (-3x^2)\, e^{-x^3}\,dx$$

$$= -2\pi \left[e^{-x^3}\right]_0^1$$

$$= -2\pi \left[e^{-1} - 1\right]$$

$$= \boxed{2\pi\left(1 - \frac{1}{e}\right)}$$

(b)[3] The same region as in part (a) is rotated about the horizontal line $y = -1$. Set up BUT DO NOT EVALUATE an integral representing the volume of the resulting solid.



By disk method:

$$V = \int_0^1 \pi\left[\left(3xe^{-x^3}+1\right)^2 - 1^2\right]dx$$