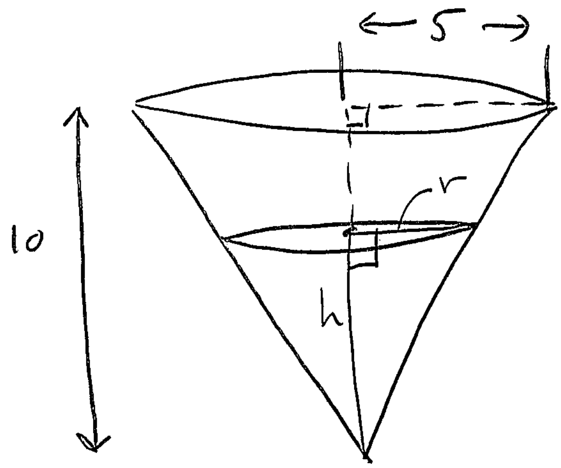
Question 1 [10 points]: An inverted cone full of water has top diameter and height both equal to 10 cm. The water drains from the cone at a rate of 32 cm^3/min. At what rate is the water level dropping when the water level in the cone is 4 cm?
(Recall: The volume of a cone of height h and base radius r is $V = \pi r^2 h/3$.)



$$\frac{dV}{dt} = -32 \ \frac{\text{cm}^3}{\text{min}}$$

Find $\frac{dh}{dt}$ when $h = 4$.

By similar triangles : $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{1}{2}h$

$\therefore V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12} h^3$

$$\therefore \frac{dV}{dt} = \frac{d}{dt}\left[\frac{\pi}{12}h^3\right]$$
$$= \frac{\pi}{12}\, 3h^2 \frac{dh}{dt}$$
$$= \frac{\pi}{4} h^2 \frac{dh}{dt}$$

When $h = 4$: $\quad -32 = \frac{\pi}{4} \cdot 4^2 \cdot \frac{dh}{dt}$

$$\therefore \frac{dh}{dt} = -\frac{8}{\pi} \ \frac{\text{cm}}{\text{min}}$$

$\boxed{\therefore \text{Water level is dropping at } \frac{8}{\pi} \ \frac{\text{cm}}{\text{min}}}$

Question 2:

(a)[5] Use a linear approximation to estimate the value of ln (0.99). (Recall that ln(1) is a "nice" number.)

Here $f(x) = \ln(x)$, $a = 1$

$f(a) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$; $f'(a) = \frac{1}{1} = 1$

$\therefore$ $f(x) \approx f(a) + f'(a)(x-a)$

$\ln(x) \approx 0 + 1 \cdot (x-1)$,

$\therefore$ $\ln(0.99) \approx 0.99 - 1 = \boxed{-0.01}$

(b)[5] Determine the linearization $L(x)$ of $f(x) = \dfrac{\tan(x) - x}{x}$ at $a = \pi$.

$f(a) = f(\pi) = \dfrac{\tan(\pi) - \pi}{\pi} = -1$

$f'(x) = \dfrac{x(\sec^2(x) - 1) - (\tan(x) - x)}{x^2}$

$f'(a) = f'(\pi) = \dfrac{\pi(\overbrace{\cancel{\sec^2(\pi)}}^{1} - 1) - (\overbrace{\cancel{\tan(\pi)}}^{0} - \pi)}{\pi^2}$

$= \dfrac{1}{\pi}$

$\therefore$ $L(x) = f(a) + f'(a)(x-a)$

$= \boxed{-1 + \frac{1}{\pi}(x - \pi)}$.

Question 3: Determine the derivative of each of the following functions (it is not necessary to simplify final answers):

(a)[3] $y = 2^{\sec x} - \dfrac{e^{-x^3}}{x}$

$$y' = 2^{\sec x} \cdot \ln(2) \cdot \sec x \tan x - \frac{x e^{-x^3}(-3x^2) - e^{-x^3}}{x^2}$$

$$= 2^{\sec x} \cdot \ln(2) \cdot \sec x \cdot \tan x + \frac{e^{-x^3}(3x^3+1)}{x^2}$$

(b)[3] $f(x) = \ln(x \sin^2 x)$

$$f'(x) = \frac{1}{x \sin^2 x} \cdot \left[\sin^2 x + x \cdot 2 \sin x \cos x\right]$$

$$= \frac{\sin x + 2x \cos x}{x \sin x}$$

(c)[4] $y = (\sqrt{x})^{x+1}$ (logarithmic differentiation may be helpful here.)

$$\ln(y) = (x+1)\ln\left(x^{\frac{1}{2}}\right)$$

$$\therefore \ln(y) = \left(\frac{x+1}{2}\right)\ln(x)$$

$$\frac{1}{y} y' = \frac{1}{2}\ln(x) + \left(\frac{x+1}{2}\right)\left(\frac{1}{x}\right)$$

$$\therefore y' = (\sqrt{x})^{x+1}\left[\frac{\ln(x)}{2} + \frac{x+1}{2x}\right].$$

Question 4: For this question use the function $f(x) = (x^2 - 8)e^x$. } domain $(-\infty, \infty)$.

(a)[7] Determine the intervals of increase and decrease of $f(x)$. State a clear conclusion.

$$f'(x) = 2x e^x + (x^2-8)e^x$$
$$= e^x[x^2+2x-8]$$
$$= e^x(x+4)(x-2)$$

$f'(x)=0$? $x=2$, $x=-4$

$f'(x)$ not exist? no such x.

		-4		2	
test:	-5		0		3
$f'(x)=e^x(x+4)(x-2)$:	$+$	0	$-$	0	$+$
$f(x)=(x^2-8)e^x$:	↗	$8e^{-4}$	↘	$-4e^2$	↗

$\therefore$ f is increasing on $(-\infty,-4) \cup (2,\infty)$.
f is decreasing on $(-4,2)$.

(b)[3] State the relative (or local) extreme values of $f(x)$.

f has a rel. max. of $8e^{-4}$ at $x=-4$;
f has a rel. min. of $-4e^2$ at $x=1$.

Question 5: Suppose $f(x)$ has domain all real numbers and first derivative $f'(x) = \dfrac{x}{x^2+36}$.

(a)[7] Determine the intervals of concavity of the graph of $y = f(x)$. State a clear conclusion.

$$f''(x) = \frac{x^2+36 - x(2x)}{(x^2+36)^2}$$

$$= \frac{36-x^2}{(x^2+36)^2}$$

$$= \frac{(6-x)(6+x)}{(x^2+36)^2}.$$

- $f''(x) = 0$? $x = 6, -6$.
- $f''(x)$ not exist? no such x.

		-6		6	
test:	-7		0		7
$f''(x)$:	$-$	0	$+$	0	$-$
$f(x)$:	$\frown$	$f(-6)$	$\smile$	$f(6)$	$\frown$

$\therefore$ Graph of f is concave down on $(-\infty,-6)\cup(6,\infty)$; concave up on $(-6,6)$.

(b)[3] State the x-coordinates of the inflection points, if any. (note: you do not have enough information to give the y-coordinates.)

$x = -6, 6.$