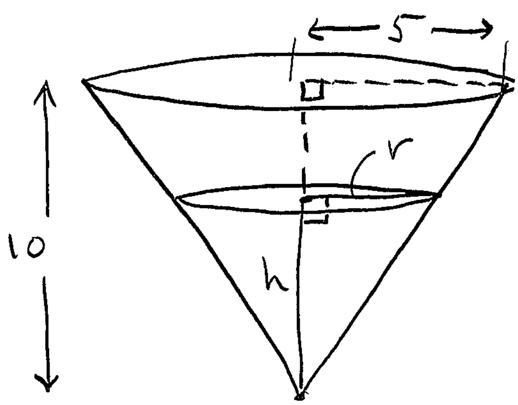
**Question 1 [10 points]:** An inverted cone full of water has top diameter and height both equal to 10 cm. The water drains from the cone at a rate of 36 $cm^3$/min. At what rate is the water level dropping when the water level in the cone is 6 cm?
(Recall: The volume of a cone of height $h$ and base radius $r$ is $V = \pi r^2 h/3$ .)




$$\frac{dV}{dt} = -36 \ \frac{cm^3}{min}$$

Find $\frac{dh}{dt}$ when $h = 6$ cm.

By similar triangles: $\frac{10}{h} = \frac{5}{r} \Rightarrow r = \left(\frac{5}{10}\right)h = \frac{1}{2}h$

$\therefore V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12} h^3$.

$$\therefore \frac{dV}{dt} = \frac{d}{dt}\left[\frac{\pi}{12} h^3\right]$$

$$= \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt}$$

$$= \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

When $h = 6$:

$$-\cancel{36} = \frac{\pi}{4} \cdot \cancel{6^2} \cdot \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{-4}{\pi} \ \frac{cm}{min}$$

$\boxed{\therefore \text{Water level is dropping at } \frac{4}{\pi} \ \frac{cm}{min}.}$

**Question 2:**

**(a)[5]** Determine the linearization $L(x)$ of $f(x) = \dfrac{x - \tan(x)}{x}$ at $a = \pi$.

$$f(a) = f(\pi) = \frac{\pi - \cancel{\tan(\pi)}^{\to 0}}{\pi} = 1$$

$$f'(x) = \frac{x(1-\sec^2 x) - (x - \tan(x))\cdot 1}{x^2}$$

$$f'(a) = f'(\pi) = \frac{\pi(1-\sec^2\pi) - (\pi - \cancel{\tan(\pi)}^{\to 0})}{\pi^2}$$

$$= \frac{\cancel{\pi(1-1)}^{\to 0} - (\pi)}{\pi^2}$$

$$= -\frac{1}{\pi}$$

$$\therefore\ L(x) = f(a) + f'(a)(x-a)$$

$$= \boxed{1 - \frac{1}{\pi}(x - \pi)}$$

**(b)[5]** Use a linear approximation to estimate the value of ln (0.9). (Recall that ln(1) is a "nice" number.)

Here $f(x) = \ln(x)$, $a = 1$.

$f(a) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$ ; $f'(a) = \frac{1}{1} = 1$

$$\therefore\ f(x) \approx f(a) + f'(a)(x-a)$$

$$\ln(x) \approx 0 + 1\cdot(x-1)$$

$$\therefore\ \ln(0.9) \approx 0.9 - 1 = \boxed{-0.1}$$

**Question 3:** Determine the derivative of each of the following functions (it is not necessary to simplify final answers):

**(a)[3]** $f(x) = \ln(x \sin^2 x)$

$$f'(x) = \frac{1}{x\sin^2 x} \cdot \left[1 \cdot \sin^2 x + x \cdot 2 \sin x \cos x\right]$$

$$= \frac{\sin x + 2x \cos x}{x \sin x}$$

**(b)[3]** $y = 2^{\sec x} - \dfrac{e^{-x^3}}{x}$

$$y' = 2^{\sec x} \cdot \ln(2) \cdot \sec x \tan x - \frac{x e^{-x^3} \cdot (-3x^2) - e^{-x^3}}{x^2}$$

**(c)[4]** $y = (\sqrt{x})^{x+1}$ (logarithmic differentiation may be helpful here.)

$$\ln(y) = (x+1) \ln(x^{1/2})$$

$$\therefore \quad \ln(y) = \left(\frac{x+1}{2}\right) \ln(x)$$

$$\frac{1}{y} y' = \frac{1}{2} \ln(x) + \left(\frac{x+1}{2}\right)\left(\frac{1}{x}\right)$$

$$\therefore \; y' = (\sqrt{x})^{x+1} \left[\frac{\ln(x)}{2} + \frac{x+1}{2x}\right].$$

**Question 4:** For this question use the function $f(x) = (x^2 - 3)e^x$ . $\}$ domain $(-\infty, \infty)$.

**(a)[7]** Determine the intervals of increase and decrease of $f(x)$. State a clear conclusion.

$$f'(x) = 2xe^x + (x^2-3)e^x$$
$$= e^x\left[x^2+2x-3\right]$$
$$= e^x(x+3)(x-1)$$

$f'(x)=0$ ? $x=-3$, $x=1$

$f'(x)$ not exist? no such $x$.

| | | -3 | | 1 | |
|---|---|---|---|---|---|
| test: | -4 | | 0 | | 2 |
| $f'(x) = e^x(x+3)(x-1)$: | + | 0 | − | 0 | + |
| $f(x) = (x^2-3)e^x$ | ↗ | $6e^{-3}$ | ↘ | $-2e$ | ↗ |

$\therefore$ $f$ is increasing on $(-\infty, -3) \cup (1, \infty)$.
$f$ is decreasing on $(-3, 1)$.

**(b)[3]** State the relative (or local) extreme values of $f(x)$.

$f$ has a rel. max. of $6e^{-3}$ at $x=-3$.

$f$ has a rel. min. of $-2e$ at $x=1$.

**Question 5:** This question deals with a function $f(x)$ for which $f'(x) = \dfrac{x}{x^2+25}$.

**(a)[7]** Determine the intervals of concavity of the graph of $y = f(x)$. State a clear conclusion.

$$f''(x) = \frac{(x^2+25)(1) - x(2x)}{(x^2+25)^2}$$

$$= \frac{25 - x^2}{(x^2+25)^2}$$

$$= \frac{(5-x)(5+x)}{(x^2+25)^2}$$

- $\underline{f''(x) = 0}$ ? $x = 5, -5$
- $\underline{f''(x) \text{ not exist}}$? no such $x$.

| | | | | | |
|---|---|---|---|---|---|
| | | $-5$ | | $5$ | |
| test : | $-6$ | | $0$ | | $6$ |
| $f''(x)$ : | $-$ | $0$ | $+$ | $0$ | $-$ |
| $f(x)$ : | $\frown$ | $f(-5)$ | $\smile$ | $f(5)$ | $\frown$ |

$\therefore$ Graph is concave down on $(-\infty, -5) \cup (5, \infty)$; concave up on $(-5, 5)$.

**(b)[3]** State the $x$-coordinates of the inflection points, if any. (note: you do not have enough information to give the $y$-coordinates.)

$x = -5, \quad x = 5.$