# 1 First Derivatives and Shapes of Curves

What information does $f'(x)$ give us about the shape of the graph of $y = f(x)$?

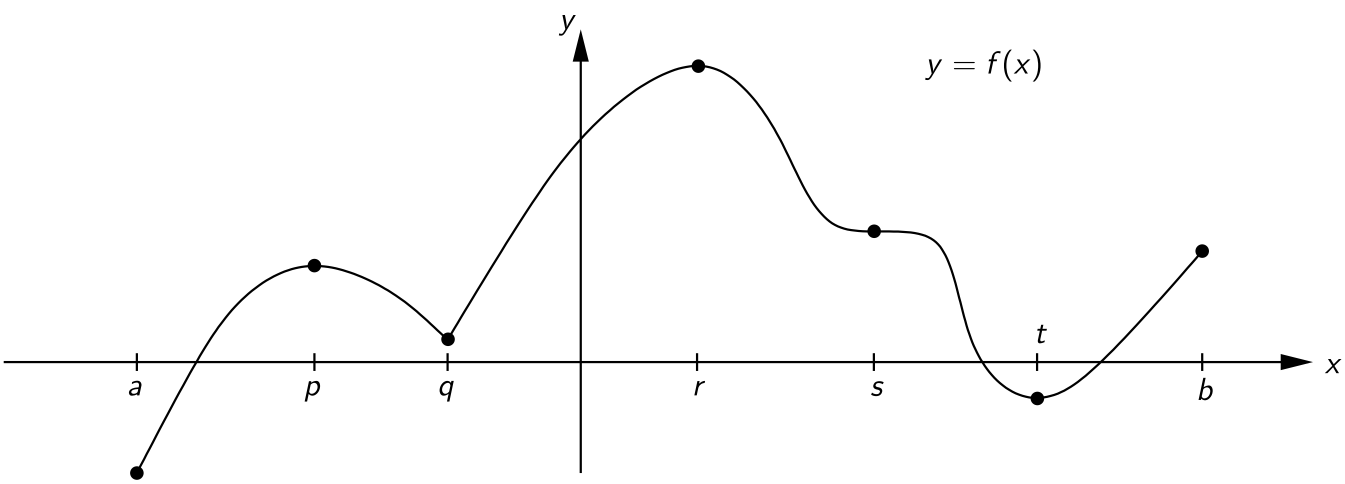
Recall: an **interval** is a continuous segment of the real line. For example, $[0, 1]$, $(-\pi, 7]$, $(0, \infty)$ and $(-\infty, \infty)$ are all intervals.

A function $f$ is said to be **increasing** on an interval if for any numbers $x_1 < x_2$ from the interval, $f(x_1) < f(x_2)$. If a function is increasing on an interval then its graph rises as $x$ increases.

Similarly, $f$ is said to be **decreasing** on an interval if for any numbers $x_1 < x_2$ from the interval, $f(x_1) > f(x_2)$. If a function is decreasing on an interval then its graph falls as $x$ increases.

Using derivatives we can easily determine the intervals of increase and decrease of a function.

Using the following general graph let's introduce some terminology and make some observations:




Here $f$ has domain $D = [a, b]$.

## 1.1 Definitions and a Theorem

**absolute (or global) maximum:** $f$ has an absolute maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every $x$ in $D$.

**absolute (or global) minimum:** $f$ has an absolute minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every $x$ in $D$.

**extreme values of $f$:** the absolute maximum of $f$ together with the absolute minimum.

**relative (or local) maximum:** $f$ has a relative maximum of $f(c)$ at $x = c$ if $f(c) \geq f(x)$ for every $x$ in an open interval containing $c$.

**relative (or local) minimum:** $f$ has a relative minimum of $f(c)$ at $x = c$ if $f(c) \leq f(x)$ for every $x$ in an open interval containing $c$.

So, referring to the graph above, we would say:

- $f$ has an absolute maximum of $f(r)$ at $x = r$;
- $f$ has an absolute minimum of $f(a)$ at $x = a$;
- $f$ has relative maxima of $f(p)$ at $x = p$ and $f(r)$ at $x = r$;
- $f$ has a relative minima of $f(q)$ at $x = q$ and $f(t)$ at $x = t$

Note:

(i) End points can correspond to absolute but not relative maxima or minima.

(ii) A point interior to the interval can correspond to both a relative and absolute maximum or minimum.

Another definition:

**critical number:** a critical number of a function $f$ is a number $c$ in the domain of $f$ such that

(i) $f'(c) = 0$, or
(ii) $f'(c)$ does not exist

Referring to our graph, $x = p$, $x = q$, $x = r$, $x = s$ and $x = t$ are critical numbers of $f$. Notice the behaviour of the graph of $y = f(x)$ at each of these critical numbers. Indeed,

**Fermat's Theorem:** If $f$ has a relative maximum or relative minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

Fermat's Theorem tells us that relative extrema must occur at critical numbers, however it does not say that every critical number corresponds to a relative extremum—look at $x = s$ in our graph above.

## 1.2 Increasing/Decreasing Test

Recall
$$f'(c) = \text{slope of the tangent line to graph of } y = f(x) \text{ at } x = c\ ,$$
so
$$\begin{aligned} f'(c) > 0 &\Rightarrow \text{outputs of } f \text{ are increasing as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is rising as } x \text{ passes through } c \\ f'(c) < 0 &\Rightarrow \text{outputs of } f \text{ are decreasing as } x \text{ passes through } c \\ &\Rightarrow \text{graph of } f \text{ is falling as } x \text{ passes through } c \end{aligned}$$

This gives the **Test for Intervals of Increase and Decrease of a Function:**

(i) If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.

(ii) If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

Observe on our graph: whenever $f$ changes from increasing to decreasing, or vice versa, it does so at a critical number ($x = p, q, r$ and $t$). However, not every critical number corresponds to such a change: $f$ is decreasing on both sides of $x = s$. Putting all of this together:

**To determine the intervals of increase and decrease of a function f:**

(i) Find points at which $f'$ changes sign (from positive to negative or vice versa). $f'$ can change sign at

- critical numbers: $x$-values at which $f'(x) = 0$ or $f'(x)$ does not exist
- values of $x$ at which $f$ itself is not defined

(ii) Test $f'(x)$ on the subintervals defined by the points from (i).

Once you have determined the intervals of increase/decrease of $f$, it is easy to read off the relative extrema (that is, the relative maxima and minima) using

**The First Derivative Test:** Suppose $x = c$ is a critical number of a continuous function $f$.

(i) If $f'$ changes from positive to negative at $x = c$, then $f$ has a relative maximum of $f(c)$ at $x = c$.

(ii) If $f'$ changes from negative to positive at $x = c$, then $f$ has a relative minimum of $f(c)$ at $x = c$.

(iii) If $f'$ does not change sign at $x = c$, then $f$ has a neither a relative maximum nor relative minimum at $x = c$.

### Example 1

Let $f(x) = 3x^{2/3} - x$.

(i) Determine the intervals of increase and decrease of $f$.

(ii) State the relative extrema.

(iii) Use the information from (i) and (ii) to sketch the graph of $y = f(x)$.