Question 1:

(a)[4] The row echelon form of a system of equations is

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Express the solution set for this system using the variables x, y and z.

$y = 2 - 3z$

$x = 4 - 2y - 4z$

$= 4 - 2(2-3z) - 4z$

$= 2z$

$\therefore$ solution set is

$\{ (2z, 2-3z, z) \mid z \text{ is any real number} \}$

(b)[4] Use matrix reduction to solve the following system of equations:

$$\begin{aligned} x - y &= 2 \\ 2x - 3y &= 2 \\ 3x - 5y &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} \textcircled{1} & -1 & 2 \\ 2 & -3 & 2 \\ 3 & -5 & 2 \end{array}\right]$$

$R_2 = (-2)r_1 + r_2$:

$R_3 = (-3)r_1 + r_3$:

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{array}\right]$$

$R_2 = (-1)r_2$:

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & \textcircled{1} & 2 \\ 0 & -2 & -4 \end{array}\right]$$

$R_1 = r_2 + r_1$:

$R_3 = 2r_2 + r_3$:

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x = 4$

$y = 2$

(c)[2] Is the system of equations in (b) consistent or inconsistent?

Consistent

Question 2: For this problem use the following matrices:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 \\ -2 & -2 \\ 5 & 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 6 & 7 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$$

(a)[4] Compute $(\mathbf{A} - 2\mathbf{C})\mathbf{B}$

$$A - 2C = \begin{bmatrix} 4 & 6 \\ -2 & -2 \\ 5 & 9 \end{bmatrix} - 2\begin{bmatrix} 5 & 0 \\ -1 & 3 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ -2 & -2 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ -2 & 6 \\ 8 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 6 \\ 0 & -8 \\ -3 & -5 \end{bmatrix}$$

$$(A-2C)B = \begin{bmatrix} -6 & 6 \\ 0 & -8 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 30 \\ -48 & -56 \\ -27 & -41 \end{bmatrix}$$

(b)[2] Suppose there is a matrix $\mathbf{D}$ such that the product $\mathbf{ADC}$ is defined. What must be the dimension (or size) of $\mathbf{D}$?

A D C

3×2 2×3 3×2

∴ D must have dimension 2×3.

(c)[2] Compute $\mathbf{I}_2\mathbf{B} - 3\mathbf{I}_2$.

$$I_2B - 3I_2 = \begin{bmatrix} -1 & 2 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 6 & 4 \end{bmatrix}$$

(d)[2] Assuming $\mathbf{B}$ is invertible, solve the following for x and y:

$$\underbrace{\begin{bmatrix} -1 & 2 \\ 6 & 7 \end{bmatrix}}_{B}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad B\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = B^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x = 0, \quad y = 0$$

Question 3:

(a)[7] Determine $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 3 \\ 1 & 0 & 6 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 3 & 2 & 1 & 0 & 0 \\ 2 & 7 & 3 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = (-2)r_1 + r_2$:

$R_3 = (-1)r_1 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & -1 & -2 & 1 & 0 \\ 0 & -3 & 4 & -1 & 0 & 1 \end{array}\right]$$

$R_1 = (-3)r_2 + r_1$:

$R_3 = (3)r_2 + r_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 7 & -3 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -7 & 3 & 1 \end{array}\right]$$

$R_1 = (-5)r_3 + r_1$:

$R_2 = r_3 + r_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 42 & -18 & -5 \\ 0 & 1 & 0 & -9 & 4 & 1 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 42 & -18 & -5 \\ -9 & 4 & 1 \\ -7 & 3 & 1 \end{bmatrix}$$

(b)[3] Use your result in part (a) to solve the following system of equations:

$$\begin{aligned} x + 3y + 2z &= 0 \\ 2x + 7y + 3z &= 1 \\ x + 6z &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & 3 \\ 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 42 & -18 & -5 \\ -9 & 4 & 1 \\ -7 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -13 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore x = -13$

$y = 3$

$z = 2$

Question 4 [10 points]: A hobby farmer wishes to raise goats and pigs. He wants to raise at most 10 goats and no more than 16 animals in total. A goat costs \$25 to raise and produces a profit of \$14, while each pig costs \$75 to raise and produces a profit of \$40. The farmer has \$900 available for the project. Determine the number of each animal that should be raised to generate the maximum possible profit.

Graph paper is provided on the next page. Carefully set up the problem, identify your variables, neatly sketch any required graphs and state a clear conclusion.

Let x = # goats
y = # pigs.

Maximize $z = 14x + 40y$
Subject to $x \leq 10$

$x + y \leq 16$

$25x + 75y \leq 900 \Rightarrow x + 3y \leq 36.$

$x \geq 0$

$y \geq 0$

Corner Points:
- By inspection: $(0,0)$, $(10,0)$, $(0,12)$
- Solving:

$x + y = 16$ ① $\quad x + 3y = 36$ ②

① $\Rightarrow x = 16 - y$

② $\Rightarrow 16 - y + 3y = 36$

$\therefore 2y = 20$

$y = 10$

$\therefore x = 16 - 10 = 6$

$\Rightarrow (6,10)$

$x + y = 16$, $x = 10$ $\Rightarrow \therefore y = 6$ $\quad \therefore (10,6)$

C.P.	$z = 14x + 40y$
$(0,0)$	0
$(10,0)$	140
$(0,12)$	480
$(6,10)$	484 ← max.
$(10,6)$	380

$\therefore$ 6 goats and 10 pigs should be raised to maximize profit

Question 4 (continued)