Question 1:

(a)[6] Given that $y_1(x) = x$ is one solution of

$$x^2y'' + 2xy' - 2y = 0, \qquad x > 0,$$

use reduction of order to find a second linearly independent solution $y_2(x)$. (Do not use "the formula", but rather, find $y_2(x)$ from first principles. You may check your result using the formula however.)

$$y_2 = uy_1 = ux$$

$$y_2' = u'x + u$$

$$y_2'' = u''x + 2u'$$

$$x^2[u''x + 2u'] + 2x[u'x + u] - 2[ux] = 0$$

$$u''x^3 + 2u'x^2 + 2u'x^2 + \cancel{2ux} - \cancel{2ux} = 0$$

$$u''x^3 + 4u'x^2 = 0 \quad \Big\} \text{ let } w = u', \; w' = u''$$

$$\frac{dw}{dx}x^3 + 4wx^2 = 0$$

$$\int \frac{1}{w}\,dw = \int -\frac{4}{x}\,dx$$

$$\ln|w| = -4\ln|x| + C_1 = \ln x^{-4} + C_1$$

$$\therefore\; w = c_2 x^{-4}$$

$$\therefore\; u' = c_2 x^{-4}$$

$$u = c_3 x^{-3} + c_4 \quad \Big\} \text{ select } c_3 = 1,\; c_4 = 0$$

$$\therefore\; y_2(x) = (x^{-3})\,y_1(x) = (x^{-3})(x) = \boxed{x^{-2}}$$

(b)[4] Compute the Wronskian $W(y_1, y_2)$ of the solution functions from part (a). What property of the Wronskian confirms that y_1 and y_2 are linearly independent on the interval $(0, \infty)$?

$$W(x, x^{-2}) = \begin{vmatrix} x & x^{-2} \\ 1 & -2x^{-3} \end{vmatrix}$$

$$= -2x^{-2} - x^{-2}$$

$$= \boxed{-3x^{-2}}$$

Since $W(x, x^{-2}) \neq 0$ on $(0,\infty)$ then

$y_1(x) = x$, $y_2(x) = x^{-2}$ are lin. indep. on $(0,\infty)$.

Question 2 [10]: Solve the following initial value problem:

$$y'' + 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

$$r^2 + 4r + 5 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\therefore \; y = C_1 e^{-2t} \cos(t) + C_2 e^{-2t} \sin(t)$$

$$y(0) = 1 \Rightarrow C_1 = 1$$

$$y'(t) = -2C_1 e^{-2t} \cos(t) - C_1 e^{-2t} \sin(t) - 2C_2 e^{-2t} \sin t + C_2 e^{-2t} \cos(t)$$

$$y'(0) = 0 \Rightarrow -2C_1 + C_2 = 0$$

$$\Rightarrow -2 + C_2 = 0$$

$$\Rightarrow C_2 = 2$$

$$\boxed{\therefore \; y(t) = e^{-2t} \cos(t) + 2e^{-2t} \sin(t)}$$

Question 3 [10]: Determine the general solution:

$$y'' + 6y' + 5y = e^{-x}$$

$r^2 + 6r + 5 = 0$

$(r+5)(r+1) = 0$

$r = -5, \quad r = -1$

$\therefore \quad y_c = C_1 e^{-5x} + C_2 e^{-x}$

For y_p, try $y_p = \cancel{Ae^{-x}}$ } term of y_c so

$y_p = Axe^{-x}$

$y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x}$

$$\therefore \quad \underbrace{\left[-2Ae^{-x} + Axe^{-x}\right]}_{y_p''} + 6\underbrace{\left[Ae^{-x} - Axe^{-x}\right]}_{y_p'} + 5\underbrace{\left[Axe^{-x}\right]}_{y_p} = e^{-x}$$

$\therefore \quad 4Ae^{-x} = e^{-x}$

$\therefore \quad 4A = 1$

$A = \frac{1}{4}$

$\therefore \quad y_p = \frac{1}{4} x e^{-x}$

$\therefore \quad y = y_c + y_p$

$$\boxed{y(x) = C_1 e^{-5x} + C_2 e^{-x} + \frac{1}{4} x e^{-x}}$$

Question 4 [10]: The temperature $u(r)$ between concentric circles of radius $r = a$ and $r = b$, (where $a < b$) is determined by the boundary value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0, \qquad u(a) = u_0, \quad u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.

Let $y = u'$, $y' = u''$:

$$r\frac{dy}{dr} + y = 0$$

$$\int \frac{1}{y}\,dy = \int -\frac{1}{r}\,dr$$

$$\ln|y| = -\ln|r| + C_1 = \ln|r|^{-1} + C_1 \quad \} \text{ note } r>0$$

$$y = C_2 r^{-1}$$

$$\therefore\ u' = \frac{C_2}{r}$$

$$u = C_2 \ln r + C_3$$

$$u(a) = u_0 \Rightarrow C_2\ln(a) + C_3 = u_0 \quad ①$$

$$u(b) = u_1 \Rightarrow C_2\ln(b) + C_3 = u_1 \quad ②$$

$$① - ② \Rightarrow C_2[\ln(a) - \ln(b)] = u_0 - u_1$$

$$\Rightarrow C_2 = \frac{u_0 - u_1}{\ln\left(\frac{a}{b}\right)}$$

$$\therefore ① \Rightarrow C_3 = u_0 - C_2\ln(a)$$

$$= u_0 - \frac{u_0 - u_1}{\ln\left(\frac{a}{b}\right)}\ln(a)$$

$$= \frac{\cancel{u_0\ln(a)} - u_0\ln(b) - \cancel{u_0\ln(a)} + u_1\ln(a)}{\ln\left(\frac{a}{b}\right)}$$

$$= \frac{u_1\ln(a) - u_0\ln(b)}{\ln\left(\frac{a}{b}\right)}$$

$$\boxed{\therefore\ u(r) = \left[\frac{u_0 - u_1}{\ln\left(\frac{a}{b}\right)}\right]\ln(r) + \frac{u_1\ln(a) - u_0\ln(b)}{\ln\left(\frac{a}{b}\right)}}$$

Question 5 [10]: A 1 kg mass is attached to a spring whose constant is $k = 16$ N/m. The entire spring-mass system is then submerged in a liquid that imparts a damping force of $\beta = 10$ times the velocity of the mass. At time $t = 0$ the mass is released from a point 1 m below equilibrium with an initial velocity of 12 m/s upward. Set up and solve the differential equation describing the position $x(t)$ of the mass for $t \geq 0$.

$$m x'' = -kx' - \beta x', \quad x(0) = 1, \quad x'(0) = -12$$

$$x'' = -16x - 10x'$$

$$x'' + 10x' + 16x = 0$$

$$r^2 + 10r + 16 = 0$$

$$(r+8)(r+2) = 0$$

$$r = -8, \quad r = -2$$

$$\therefore \quad x(t) = c_1 e^{-8t} + c_2 e^{-2t}$$

$$x(0) = 1 \Rightarrow c_1 + c_2 = 1$$

$$x'(t) = -8c_1 e^{-8t} - 2c_2 e^{-2t}$$

$$x'(0) = -12 \Rightarrow -8c_1 - 2c_2 = -12$$

$$c_1 + c_2 = 1 \quad \ldots (1)$$

$$-8c_1 - 2c_2 = -12 \quad \ldots (2)$$

$$(1) \Rightarrow c_1 = 1 - c_2$$

$$(2) \Rightarrow -8(1-c_2) - 2c_2 = -12$$

$$-8 + 8c_2 - 2c_2 = -12$$

$$6c_2 = -4$$

$$c_2 = -\frac{2}{3}$$

$$\therefore \quad c_1 = 1 - c_2 = \frac{5}{3}$$

$$\boxed{\therefore \quad x(t) = \frac{5}{3} e^{-8t} - \frac{2}{3} e^{-2t}}$$

Question 6 [10]: Determine the general solution $\mathbf{X}(t)$ of the following system. Express your final answer in a form that does not contain complex numbers.

$$\mathbf{X}'(t) = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{X}(t)$$

Eigenvalues:

$$\begin{vmatrix} 5-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(3-\lambda)+2=0$$

$$\lambda^2 - 8\lambda + 17 = 0$$

$$\lambda = \frac{8 \pm \sqrt{64-4(1)(17)}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$$

Eigenvector for $\lambda = 4+i$:

$$\left[\begin{array}{cc|c} 5-(4+i) & 1 & 0 \\ -2 & 3-(4+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1-i & 1 & 0 \\ -2 & -1-i & 0 \end{array}\right]$$

$r_1 \leftrightarrow r_2$: $\left[\begin{array}{cc|c} -2 & -1-i & 0 \\ 1-i & 1 & 0 \end{array}\right]$

$R_1 = -\frac{1}{2} r_1$: $\left[\begin{array}{cc|c} 1 & +\frac{1}{2}+\frac{i}{2} & 0 \\ 1-i & 1 & 0 \end{array}\right]$

$R_2 = -(1-i)r_1 + r_2$: $\left[\begin{array}{cc|c} 1 & \frac{1}{2}+\frac{i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore k_1 + \left(\frac{1}{2}+\frac{i}{2}\right)k_2 = 0$

$\therefore K_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$ upon selecting $k_2 = 2$.

$\therefore$ a solution is $X_1(t) = \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(4+i)t}$

$$= e^{4t} \begin{bmatrix} 1+i \\ 2 \end{bmatrix} (\cos(t) + i\sin(t))$$

$$= e^{4t} \begin{bmatrix} \cos(t) - \sin(t) \\ 2\cos(t) \end{bmatrix} + i e^{4t} \begin{bmatrix} \cos(t) + \sin(t) \\ 2\sin(t) \end{bmatrix}$$

$\therefore$ General sol$^{\underline{n}}$ is

$$X(t) = C_1 e^{4t} \begin{bmatrix} \cos(t) - \sin(t) \\ 2\cos(t) \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} \cos(t) + \sin(t) \\ 2\sin(t) \end{bmatrix}$$