Question 1:

(a)[5] Solve the following IVP:

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}, \qquad y(\pi/2) = 1$$

$$\int \frac{1+2y^2}{y}\,dy = \int \cos x\,dx$$

$$\int \frac{1}{y} + 2y\,dy = \int \cos x\,dx$$

$$\ln|y| + y^2 = \sin x + C$$

$$y\left(\tfrac{\pi}{2}\right) = 1 \Rightarrow \cancel{\ln|1|}^{\,0} + 1 = \cancel{\sin\left(\tfrac{\pi}{2}\right)}^{\,1} + C$$

$$\therefore\ C = 0$$

$$\boxed{\therefore\ \ln|y| + y^2 = \sin x}$$

(b)[5] Solve the following IVP:

$$y' + 2y = xe^{-2x}, \qquad y(1) = 0$$

$$e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$$

$$\therefore\ \frac{d}{dx}\left[e^{2x}y\right] = e^{2x}xe^{-2x}$$

$$\Rightarrow\ e^{2x}y = \int x\,dx = \frac{x^2}{2} + C$$

$$\Rightarrow\ y = \frac{x^2}{2}e^{-2x} + Ce^{-2x}$$

$$y(1) = 0 \Rightarrow\ 0 = \frac{1}{2}e^{-2} + Ce^{-2}$$

$$\Rightarrow\ C = \frac{-1}{2}$$

$$\therefore\ \boxed{y = \frac{x^2}{2}e^{-2x} - \frac{1}{2}e^{-2x}}$$

Question 2:

(a)[5] Solve the following differential equation:

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0$$

$$\underbrace{(y\cos x + 2xe^y)}_{M}dx + \underbrace{(\sin x + x^2e^y - 1)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y = \frac{\partial N}{\partial x}, \text{ so exact.}$$

$$\therefore f = \int y\cos x + 2xe^y\,dx = y\sin x + x^2e^y + g(y)$$

$$\frac{\partial f}{\partial y} = \sin x + x^2e^y + g'(y) = \sin x + x^2e^y - 1$$

$$\therefore g'(y) = -1 \Rightarrow g(y) = -y$$

$$\therefore f(x,y) = y\sin x + x^2e^y - y$$

$$\therefore \text{sol}^n \text{ is } \boxed{y\sin x + x^2e^y - y = C}$$

(b)[5] Find a non-trivial solution to the following differential equation:

$$x^2\frac{dy}{dx} - 2xy = 3y^4$$

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{3}{x^2}y^4 \quad \Big\} \text{ Bernoulli, } n=4$$

Let $u = y^{1-4} = y^{-3}$, so $y = u^{-1/3}$, $\frac{dy}{dx} = -\frac{1}{3}u^{-4/3}\frac{du}{dx}$

$\therefore$ equation is $-\frac{1}{3}u^{-4/3}\frac{du}{dx} - \frac{2}{x}u^{-1/3} = \frac{3}{x^2}u^{-4/3}$

$$\Rightarrow \frac{du}{dx} + \frac{6}{x}u = -\frac{9}{x^2}$$

$$e^{\int P(x)\,dx} = e^{\int \frac{6}{x}dx} = e^{6\ln|x|} = x^6$$

$$\therefore \frac{d}{dx}\left[x^6u\right] = \left(-\frac{9}{x^2}\right)(x^6) \Rightarrow x^6u = \int -9x^4\,dx = -\frac{9}{5}x^5 + C$$

$$\therefore u = -\frac{9}{5}\frac{1}{x} + \frac{C}{x^6}$$

$$\therefore \boxed{y^{-3} = -\frac{9}{5}\frac{1}{x} + \frac{C}{x^6}}$$

Question 3: For this question consider the differential equation

$$\frac{dy}{dx} = y\ln(1+y^2) - y\ln 5$$

(a)[2] Determine the equilibrium solution(s).

$$y\ln(1+y^2) - y\ln(5) = 0$$

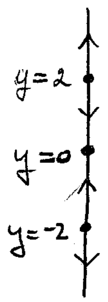
$$\Rightarrow \quad y\left[\ln\left(\frac{1+y^2}{5}\right)\right] = 0$$

$$\Rightarrow \quad y = 0, \quad \frac{1+y^2}{5} = 0$$

$$\Rightarrow \quad y = 0, \quad y^2 = 4$$

$$\Rightarrow \quad y = 0, \quad y = 2, \quad y = -2$$

(b)[2] Sketch the one-dimensional phase portrait.



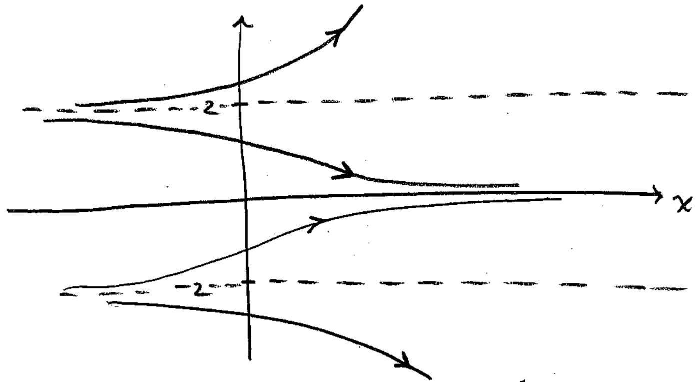
$$\frac{dy}{dx} = y\left[\ln\left(\frac{1+y^2}{5}\right)\right]$$

(c)[2] Classify each critical point.

$y = 2$ and $y = -2$ are both unstable

$y = 0$ is asymptotically stable.

(d)[2] Graph the equilibrium solutions and sketch typical solution curves in the regions between the equilibria.

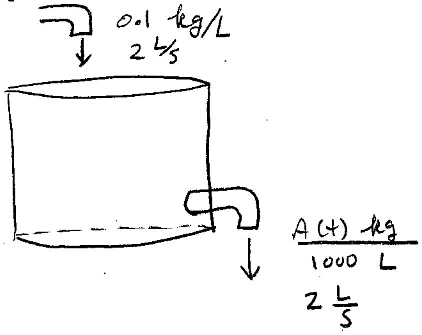


(e)[2] Suppose we are given the initial condition $y(100) = \frac{1}{e}$. What is $\lim_{x\to\infty} y(x)$?

Since $0 < y(100) < 2$, using (d) $\lim_{x\to\infty} y(x) = 0$

Question 4: A tank contains 50 kg of salt dissolved in 1000 L of water. Water with a concentration of 0.1 kg/L begins entering the tank at a rate of 2 L/s, while the well-mixed solution is pumped out the bottom of the tank at the same rate.

(a)[2] Write the IVP which models the amount $A(t)$ (in kg) of dissolved salt in the tank at time t.



$$\therefore \frac{dA}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(0.1\ \frac{kg}{L}\right)\left(2\ \frac{L}{s}\right) - \left(\frac{A\ kg}{1000\ L}\right)\left(2\ \frac{L}{s}\right)$$

$$\boxed{\frac{dA}{dt} = 0.2 - 0.002\,A \quad \frac{kg}{s}}$$

Initially: $A(0) = 50$ kg.

(b)[6] Determine how much dissolved salt is in the tank after five minutes.

Using (a) : $\displaystyle\int \frac{1}{0.2 - 0.002A}\, dA = \int dt$

$$-\frac{1}{0.002} \ln\left|0.2 - 0.002A\right| = t + C_1$$

$$\ln\left|0.2 - 0.002A\right| = -0.002t + C_2$$

$$0.2 - 0.002A = C_3 e^{-0.002t}$$

$$\therefore A = \frac{0.2}{0.002} + C_4 e^{-0.002t}$$

$$\therefore A(t) = 100 + C_4 e^{-0.002t}$$

$$A(0) = 50 \Rightarrow 50 = 100 + C_4$$

$$\therefore C_4 = -50$$

$$\therefore A(t) = 100 - 50e^{-0.002t}$$

$\therefore$ When $t = (5)(60) = 300$ s : $A(300) = 100 - 50e^{-(0.002)(300)}$

$$\approx \boxed{72.6\ kg}$$

(c)[2] Is there a time t at which the amount of salt in the tank will be double its initial value? Explain.

No. $A(0) = 50$ kg, and as t increases, $A(t)$ increases and asymptotically approaches 100 kg, but never reaches 100 kg in finite time.

Question 5: For this question use the IVP $y' = 0.5 - x + 2y$, $y(0) = 1$.

(a)[4] Use Euler's Method with step size $h = 0.1$ to approximate $y(0.2)$ to three decimals.

n	x_n	y_n	$y_{n+1} = y_n + h[0.5 - x_n + 2y_n]$
0	0	1	1.25
1	0.1	1.25	1.54
2	0.2	1.54	

$\therefore\ y(0.2) \approx y_2 = 1.54$

(b)[4] Solve the IVP to determine the actual solution.

$$\frac{dy}{dx} - 2y = 0.5 - x$$

$$e^{\int P(x)\,dx} = e^{\int -2\,dx} = e^{-2x}$$

$$\therefore \frac{d}{dx}\left[e^{-2x}y\right] = 0.5e^{-2x} - xe^{-2x}$$

$$\therefore e^{-2x}y = \int 0.5e^{-2x} - xe^{-2x}\,dx$$

$$= -\frac{1}{4}e^{-2x} - \int x\,d\left[\frac{e^{-2x}}{-2}\right]$$

$$= -\frac{1}{4}e^{-2x} + \frac{x}{2}e^{-2x} + \int \frac{e^{-2x}}{-2}\,dx$$

$$= \cancel{-\frac{1}{4}e^{-2x}} + \frac{x}{2}e^{-2x} + \cancel{\frac{1}{4}e^{-2x}} + C$$

$$\therefore y = \frac{x}{2} + Ce^{2x}. \quad y(0) = 1 \Rightarrow C = 1 \quad \boxed{\therefore y = \frac{x}{2} + e^{2x}}$$

(c)[2] Determine the relative error in your approximation (a).

Using (b), $y(0.2) = \frac{0.2}{2} + e^{2(0.2)} = 0.1 + e^{0.4}$

$$\therefore E_{rel} = \left|\frac{0.1 + e^{0.4} - 1.54}{0.1 + e^{0.4}}\right| \approx \boxed{0.0326 \text{ or } 3.26\%}$$

(d)[2 bonus marks] Suppose you don't know the actual solution of the IVP. Can you think of a way of using the IVP itself to determine if your approximation in (a) is an over or under-estimate?

Using $y' = 0.5 - x + 2y$

$\Rightarrow \frac{d}{dx}[y] = \frac{d}{dx}[0.5 - x + 2y]$

$y'' = -1 + 2y'$

$= -1 + 2[0.5 - x + 2y]$

$\therefore y'' = -2x + 4y$

$\therefore$ near $(x, y) = (0, 1)$, $y'' > 0$, so curve is <u>concave up</u>.

$\therefore$ tangent lines corresponding to Euler's Method lie <u>below</u> solution curve, resulting in <u>underestimates</u>.