Question 1:

(a)[6] Given that $y_1(x) = 1/x$ is one solution of

$$2x^2y'' + 3xy' - y = 0, \qquad x > 0, \quad \Big\} \; *$$

use reduction of order to find a second linearly independent solution $y_2(x)$. (Do not use "the formula", but rather, find $y_2(x)$ from first principles. You may check your result using the formula however.)

Let $y = u(x)\, x^{-1}$, $y' = u'x^{-1} - ux^{-2}$, $y'' = u''x^{-1} - 2u'x^{-2} + 2ux^{-3}$.

Sub. into $*$:

$$2x^2\left[u''x^{-1} - 2u'x^{-2} + 2ux^{-3}\right] + 3x\left[u'x^{-1} - ux^{-2}\right] - \left[ux^{-1}\right] = 0$$

$$2xu'' + (-4+3)u' + (4x^{-1} - 3x^{-1} - x^{-1})u = 0$$

$$2xu'' - u' = 0 \quad \Big\} \text{ let } w = u',\ w' = u''$$

$$2xw' - w = 0$$

$$\frac{w'}{w} = \frac{1}{2x}$$

$$\int \frac{w'}{w}\, dw = \int \frac{1}{2x}\, dx$$

$$\ln|w| = \frac{1}{2}\ln|x| \quad \text{note } x>0$$

$$w = c_1 x^{1/2}$$

$$\therefore u' = c_1 x^{1/2}$$

$$u = c_2 x^{3/2} + c_3$$

$$\therefore y_2 = uy_1 = (x^{3/2})(x^{-1}) = \boxed{x^{1/2}} \quad \text{selecting } c_2 = 1,\ c_3 = 0$$

(b)[4] Show that $y_1(x)$ and $y_2(x)$ from part (a) are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{x} & x^{1/2} \\ \frac{-1}{x^2} & \frac{1}{2}x^{-1/2} \end{vmatrix}$$

$$= \frac{1}{2}x^{-3/2} + x^{-3/2}$$

$$= \frac{3}{2}x^{-3/2}$$

$$\neq 0 \text{ on } (0,\infty).$$

Question 2:

(a)[5] Solve the following initial value problem:

$$y'' - 4y' + 4y = 0, \qquad y(0) = 12, y'(0) = -3$$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$\therefore \ y = c_1 e^{2t} + c_2 t e^{2t}$

$y(0) = 12 \Rightarrow c_1 = 12 \Rightarrow y = 12e^{2t} + c_2 t e^{2t}$

$y'(0) = -3 \Rightarrow 24e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}\Big|_{t=0} = -3$

$\Rightarrow 24 + c_2 = -3$

$\Rightarrow c_2 = -27$

$$\boxed{\therefore \ y = 12e^{2t} - 27te^{2t}}$$

(b)[5] Find the general solution of

$$y'' + y' + y = 0$$

$r^2 + r + 1 = 0$

$r = \dfrac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$\boxed{\therefore \ y = c_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)}$$

Question 3:

(a)[5] Find the general solution of

$$y'' - 3y' - 4y = 0$$

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r = 4,\ r = -1$

$$\therefore\ \boxed{y = C_1 e^{4t} + C_2 e^{-t}}$$

(b)[5] Use part (a) to find the general solution of

$$\left. y'' - 3y' - 4y = 3xe^{2x} \right\} *$$

From (a) we have $y_c = C_1 e^{4t} + C_2 e^{-t}$.

Try $y_p = (Ax + B)e^{2x} = Axe^{2x} + Be^{2x}$

So $y_p' = Ae^{2x} + 2Axe^{2x} + 2Be^{2x} = (A + 2B)e^{2x} + 2Axe^{2x}$

$y_p'' = (2A + 4B)e^{2x} + 2Ae^{2x} + 2Axe^{2x}(2)$

$= (4A + 4B)e^{2x} + 4Axe^{2x}$.

Sub. y_p, y_p', y_p'' into $*$:

$[(4A+4B)e^{2x} + 4Axe^{2x}] - 3[(A+2B)e^{2x} + 2Axe^{2x}] - 4[Axe^{2x} + Be^{2x}] = 3xe^{2x}$

e^{2x}: $4A - 3A + 4B - 6B - 4B = 0 \Rightarrow A - 6B = 0 \Rightarrow B = \frac{1}{6}A \Rightarrow B = \frac{-1}{12}$

xe^{2x}: $4A - 6A - 4A = 3 \Rightarrow -6A = 3 \Rightarrow A = \frac{-1}{2}$

$\therefore\ y_p = \left(-\frac{1}{2}x - \frac{1}{12}\right)e^{2x}$

$\therefore\ y = y_c + y_p = \boxed{C_1 e^{4x} + C_2 e^{-x} + \left(-\frac{1}{2}x - \frac{1}{12}\right)e^{2x}}$

Question 4 [10]: The temperature $u(r)$ between concentric spheres of radius $r = a$ and $r = b$, (where $a < b$) is determined by the boundary value problem

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0, \qquad u(a) = u_0,\ u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.
(You may recognize this as homework problem 5.2.27 which was not collected for grading. As a hint, let $y = du/dr$ and reduce the problem to a separable equation.)

Let $y = u'$, so $y' = u''$:

$$ry' + 2y = 0$$

$$\frac{y'}{y} = -\frac{2}{r}$$

$$\int \frac{y'}{y}\,dy = \int \frac{-2}{r}\,dr$$

$$\ln|y| = -2\ln|r| + C_1$$

$$|y| = C_2|r|^{-2}$$

$$y = C_3 r^{-2}$$

$$u' = C_3 r^{-2}$$

$$u = C_4 r^{-1} + C_5$$

$$u(a) = u_0 \Rightarrow \frac{C_4}{a} + C_5 = u_0 \ldots ①$$

$$u(b) = u_1 \Rightarrow \frac{C_4}{b} + C_5 = u_1 \ldots ②$$

$$\therefore ① - ② \Rightarrow \frac{C_4}{a} - \frac{C_4}{b} = u_0 - u_1$$

$$C_4\left(\frac{b-a}{ab}\right) = u_0 - u_1$$

$$C_4 = ab\left(\frac{u_0-u_1}{b-a}\right)$$

$$\therefore ① \Rightarrow C_5 = u_0 - \frac{C_4}{a} = u_0 - b\left(\frac{u_0-u_1}{b-a}\right) = \frac{\cancel{bu_0} - au_0 - \cancel{bu_0} + bu_1}{b-a}$$

$$= \frac{bu_1 - au_0}{b-a}$$

$$\boxed{\therefore u(r) = ab\left(\frac{u_0-u_1}{b-a}\right)\left(\frac{1}{r}\right) + \left(\frac{bu_1-au_0}{b-a}\right)}$$

Question 5 [10]: Determine the solution of the initial value problem

$$\mathbf{X}' = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix}\mathbf{X}, \quad \mathbf{X}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} 1-\lambda & 9 \\ -1 & -5-\lambda \end{vmatrix} = (1-\lambda)(-5-\lambda)+9$
$= \lambda^2+4\lambda+4$
$= (\lambda+2)^2$
$= 0$ for $\lambda=-2$ } repeated eigenvalue.

Eigenvector: Solve $\left[\begin{array}{cc|c} 1-(-2) & 9 & 0 \\ -1 & -5-(-2) & 0 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 3 & 9 & 0 \\ -1 & -3 & 0 \end{array}\right]$

$R_1 = \frac{1}{3}r_1$: $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -1 & -3 & 0 \end{array}\right]$

$R_2 = r_1+r_2$: $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ } $\therefore k_1+3k_2=0$
selecting $k_2=1$, $k_1=-3$
$\therefore \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector

$\therefore X_1(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-2t}$ is one solution.

For second lin. indep. solution, solve $(A-\lambda I)K = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$:

$\left[\begin{array}{cc|c} 3 & 9 & -3 \\ -1 & -3 & 1 \end{array}\right]$

$R_1 = \frac{1}{3}r_1$: $\left[\begin{array}{cc|c} 1 & 3 & -1 \\ -1 & -3 & 1 \end{array}\right]$

$R_2 = r_1+r_2$: $\left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$\therefore k_1+3k_2=-1$

$k_1 = -1-3k_2$

$\therefore K = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ upon selecting $k_2=0$

$\therefore$ second solution is

$X_2(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t}$

$\therefore X(t) = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-2t} + c_2\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t}\right)$

$X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} -3c_1 - c_2 &= 1 \\ c_1 &= -1 \end{aligned}$

$\therefore c_1 = -1, \quad c_2 = -3c_1 - 1 = 2$

$\therefore$ solution is

$$\boxed{X(t) = (-1)\begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-2t} + 2\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{-2t}\right)}$$

or $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} - 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} te^{-2t}$.