Question 1:

(a)[5] Solve the following differential equation and determine the largest interval I on which the solution is defined:

$$\frac{dy}{dt} + 2(t+1)y^2 = 0, \qquad y(0) = -1/8$$

$$\int \frac{1}{y^2}\,dy = \int -2(t+1)\,dt$$

$$-\frac{1}{y} = -t^2 - 2t + C$$

$$y(0) = -\frac{1}{8} \Rightarrow \frac{-1}{\left(-\frac{1}{8}\right)} = 0 + 0 + C$$

$$\Rightarrow C = 8$$

$$\therefore -\frac{1}{y} = -t^2 - 2t + 8$$

or $y = \dfrac{1}{t^2+2t-8} = \dfrac{1}{(t-2)(t+4)}$, which is defined on $(-\infty,-4)\cup(-4,2)\cup(2,\infty)$.

Since the initial condition corresponds to $(-4,2)$,

$$\boxed{y = \frac{1}{t^2+2t-8}, \quad I = (-4,2).}$$

(b)[5] Solve the following differential equation:

$$\frac{dx}{dy} = -\frac{4y^2+6xy}{3y^2+2x}$$

$$(3y^2+2x)\,dx + (4y^2+6xy)\,dy = 0$$

$\dfrac{\partial M}{\partial y} = 6y$; $\dfrac{\partial N}{\partial x} = 6y$, so equation is exact.

$$\therefore \frac{\partial f}{\partial x} = 3y^2 + 2x \Rightarrow f = \int (3y^2+2x)\,dx = 3xy^2 + x^2 + g(y)$$

$$\therefore \frac{\partial f}{\partial y} = 6xy + g'(y) = 4y^2 + 6xy \Rightarrow g'(y) = 4y^2$$

$$\Rightarrow g(y) = \frac{4}{3}y^3 + C.$$

$$\therefore f(x,y) = 3xy^2 + x^2 + \frac{4}{3}y^3 + C$$

and solution is

$$\boxed{3xy^2 + x^2 + \frac{4}{3}y^3 + C = 0.}$$

Question 2:

(a)[5] Solve the following differential equation:

$$(x+2)^2\frac{dy}{dx} = 5 - 8y - 4xy$$
$$= 5 - 4y(x+2)$$

Rewrite : $\left.\dfrac{dy}{dx} + \left(\dfrac{4}{x+2}\right)y = \dfrac{5}{(x+2)^2}\right\}$ Linear, $P(x) = \dfrac{4}{x+2}$

$f(x) = \dfrac{5}{(x+2)^2}$

$$e^{\int P(x)dx} = e^{\int \frac{4}{x+2}dx} = e^{4\ln|x+2|} = (x+2)^4$$

$$\therefore \frac{d}{dx}\left[(x+2)^4 y\right] = 5(x+2)^2$$

$$\therefore (x+2)^4 y = \frac{5}{3}(x+2)^3 + C$$

$$\boxed{\therefore \quad y = \frac{5}{3}\left(\frac{1}{x+2}\right) + \frac{C}{(x+2)^4}}$$

(b)[5] Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

$$(x^2 - y^2)dx - 2xy\,dy = 0$$

$$\frac{\partial M}{\partial y} = -2y \;;\; \frac{\partial N}{\partial x} = -2y, \text{ so exact.}$$

$$\therefore \frac{\partial f}{\partial x} = x^2 - y^2 \Rightarrow f = \int x^2 - y^2\,dx = \frac{x^3}{3} - xy^2 + g(y)$$

$$\therefore \frac{\partial f}{\partial y} = -2xy + g'(y) = -2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

$$\therefore f(x,y) = \frac{x^3}{3} - xy^2 + C$$

and solution is $\boxed{\dfrac{x^3}{3} - xy^2 + C = 0}$

Question 3:

(a)[5] Solve the following differential equation:

$$* \quad \frac{dy}{dx} - y = e^x y^2 \quad \Bigg\} \begin{array}{l} \text{Bernoulli with} \\ P(x) = -1,\ f(x) = e^x,\ n = 2 \end{array}$$

Let $u = y^{1-2} = y^{-1}$, so $y = \frac{1}{u}$ and $\frac{dy}{dx} = \frac{-1}{u^2}\frac{du}{dx}$.

$\therefore$ $*$ becomes $\frac{-1}{u^2}\frac{du}{dx} - \frac{1}{u} = e^x \frac{1}{u^2}$

$$\Rightarrow \frac{du}{dx} + u = -e^x \quad \Big\} P(x) = 1,\ f(x) = -e^x$$

$$e^{\int P(x)dx} = e^{\int 1 dx} = e^x$$

$$\therefore \frac{d}{dx}\left[ue^x\right] = -e^{2x} \Rightarrow ue^x = \int -e^{2x}dx = -\frac{e^{2x}}{2} + C$$

$$\Rightarrow u = -\frac{e^x}{2} + C_1 e^{-x}$$

$$\Rightarrow \frac{1}{y} = -\frac{e^x}{2} + C_1 e^{-x}$$

$$\Rightarrow \boxed{y = \frac{2}{C_2 e^{-x} - e^x}}$$

(b)[5] Solve the following differential equation:

$$* \begin{cases} \frac{dy}{dx} = \cos(x+y), \quad y(0) = \pi/4 \end{cases}$$

Let $u = x + y$, $\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\therefore$ $*$ becomes $\frac{du}{dx} - 1 = \cos(u)$

$$\frac{du}{dx} = 1 + \cos(u)$$

$$\int \frac{du}{1+\cos(u)} = \int dx = x + C$$

$$\int \frac{1}{1+\cos(u)}du = \int \frac{1-\cos(u)\ du}{(1+\cos(u))(1-\cos(u))} = \int \frac{1-\cos(u)}{\sin^2(u)}du$$

$$= \int \csc^2 u\, du - \int \frac{\cos(u)}{\sin^2(u)}du$$

$$= -\cot(u) + \csc(u)$$

$\therefore$ $-\cot(u) + \csc(u) = x + C$

$$\boxed{-\cot(x+y) + \csc(x+y) = x + C}$$

Question 4: For this question consider the differential equation

$$\frac{dP}{dt} = aP\ln\left(\frac{b}{P}\right)$$

where a and b are positive constants.

(a)[4] Determine the equilibrium solution(s).

$$aP\ln\left(\frac{b}{P}\right) = 0 \Rightarrow \ln\left(\frac{b}{P}\right) = 0$$

(note: $P \neq 0$ because of $\ln\left(\frac{b}{P}\right)$ term)

$$\Rightarrow \frac{b}{P} = 1$$

$$\Rightarrow \boxed{P = b}$$

(b)[5] Solve the differential equation.

$$\int \frac{1}{P\ln\left(\frac{b}{P}\right)}\,dP = \int a\,dt$$

$$\int \frac{1}{P[\ln b - \ln P]}\,dP = at + C_1$$

$$-\ln\left(\ln\left(\frac{b}{P}\right)\right) = at + C_1$$

$$\therefore \ln\left(\frac{b}{P}\right) = C_2 e^{-at}$$

$$\frac{b}{P} = e^{C_2 e^{-at}}$$

$$\boxed{\therefore P = b e^{C_3 e^{-at}}}$$

(c)[1] What is $\lim_{t\to\infty} P(t)$?

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} b e^{C_3 e^{-at}} = be^0 = \boxed{b}$$

Question 5:

(a)[5] A fish population is decreasing at a rate that is proportional to the square root of its size. Initially there were 90,000 fish, and after six weeks 40,000 remain. Solve the corresponding differential equation to determine the time at which the population will be reduced to 10,000.

$$\frac{dP}{dt} = -k\sqrt{P}, \quad P(0) = 90{,}000, \quad P(6) = 40{,}000.$$

$$\int P^{-\frac{1}{2}}\,dP = \int k\,dt \Rightarrow 2P^{\frac{1}{2}} = kt + C$$

$$P(0) = 90{,}000 \Rightarrow 2(90000)^{\frac{1}{2}} = k \cdot 0 + C \Rightarrow C = 600.$$

$$P(6) = 40{,}000 \Rightarrow 2(40000)^{\frac{1}{2}} = k \cdot 6 + 600 \Rightarrow k = \frac{400-600}{6} = -\frac{100}{3}.$$

$$\therefore 2P^{\frac{1}{2}} = -\frac{100}{3}t + 600 \quad \text{or} \quad P = \left(-\frac{50}{3}t + 300\right)^2$$

If $P = 10{,}000$: $10000 = \left(-\frac{50}{3}t + 300\right)^2$

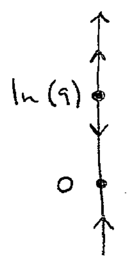
$$\Rightarrow t = \frac{3(100-300)}{-50} = \boxed{12 \text{ weeks.}}$$

(b)[5] Consider the autonomous differential equation

$$\frac{dy}{dx} = \frac{ye^y - 9y}{e^y}$$

(i) Determine the critical points and sketch the one-dimensional phase portrait.

$$\frac{ye^y - 9y}{e^y} = \frac{y(e^y - 9)}{e^y} = 0 \Rightarrow y = 0;\ e^y = 9,\ y = \ln(9)$$

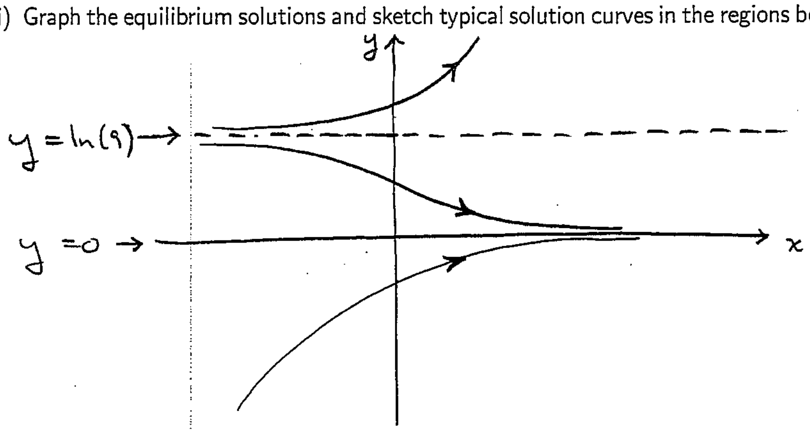


(ii) Classify each critical point as asymptotically stable, unstable, or semi-stable.

$c = \ln(9)$ is unstable.

$c = 0$ is asymptotically stable.

(iii) Graph the equilibrium solutions and sketch typical solution curves in the regions between the equilibria.



Question 6:

(a)[4] For this question use the IVP $y' = 2y\cos(x)$, $y(0) = 1$.

(i) Use Euler's Method with step size $h = 0.1$ to approximate $y(0.2)$.

n	x_n	y_n	$y_{n+1} = y_n + h\, f(x_n, y_n)$
0	0	1	1.2000
1	0.1	1.2000	1.4388
2	0.2	1.4388	

$$\therefore\ y(0.2) \approx 1.439$$

(ii) The actual solution to the IVP is $y = e^{2\sin x}$. Use this fact to determine the relative error in your approximation in part (i).

$$E_{rel} = \left| \frac{\text{actual} - \text{estimate}}{\text{actual}} \right| = \left| \frac{e^{2\sin(0.2)} - 1.439}{e^{2\sin(0.2)}} \right| \approx 0.0328 \approx 3.28\%$$

(b)[6] Write a differential equation which models each of the following situations, and state a reasonable initial condition for each. Do not solve the differential equations.

(i) The acceleration of a car is proportional to the difference between 250 km/h and the velocity of the car.

$$\frac{dv}{dt} = k(250 - v), \quad v(0) = v_0.$$

(ii) The size of an alligator population in a region follows a logistic model. Hunting is then allowed and alligators are removed from the region at a rate proportional to the existing population size.

$$\frac{dP}{dt} = k_1 P(M - P) - k_2 P, \quad P(0) = P_0.$$

(iii) In the theory of learning, the rate at which a subject is learned is assumed to be proportional to the amount that is left to be learned. Let $A(t)$ represent the amount learned by time t, and M be the total amount to be learned.

$$\frac{dA}{dt} = k(M - A), \quad A(0) = A_0$$