Question 1:

(a)[5] Determine $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = I$

$u = x^{1/2}$

$du = \frac{1}{2x^{1/2}}\,dx$

$\therefore \quad I = 2\int e^u\,du$

$= 2e^u + C$

$= \boxed{2e^{\sqrt{x}} + C}$

(b)[5] Determine $\int \sin^3 x \cos^2 x\,dx$

$= \int \sin^2 x \cos^2 x \sin x\,dx$

$= \int (1-\cos^2 x)\cos^2 x \sin x\,dx \quad \Big\} \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}$

$= -\int (1-u^2)\,u^2\,du$

$= -\int u^2 - u^4\,du$

$= -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= \boxed{-\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C}$

Question 2:

(a)[5] Compute $I = \int_1^2 \frac{\ln x}{x^2}\,dx$

$$\int (\ln x)\, x^{-2}\,dx \quad \left.\begin{matrix} u = \ln x & dv = x^{-2}dx \\ du = \frac{1}{x}dx & v = \frac{-1}{x}\end{matrix}\right\}$$

$$= (\ln x)\left(\frac{-1}{x}\right) + \int \frac{1}{x}\,\frac{1}{x}\,dx$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\therefore\ I = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^2$$

$$= \frac{-\ln 2}{2} - \frac{1}{2} + 0 + 1$$

$$= \boxed{\frac{1-\ln 2}{2}}$$

(b)[5] Determine $\int \frac{1}{x(x-1)^2}\,dx$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x^2-2x+1) + Bx^2 - Bx + Cx}{x(x-1)^2}$$

$$= \frac{(A+B)x^2 + [-2A-B+C]x + A}{x(x-1)^2}$$

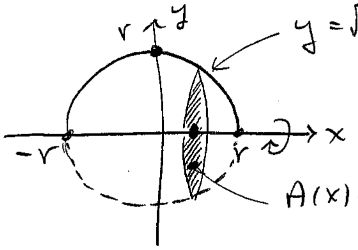
$\therefore\ A = 1$; $A+B=0 \Rightarrow B=-1$; $-2A-B+C=0 \Rightarrow C = 2-1=1$

$$\therefore \int \frac{1}{x(x-1)^2}\,dx = \int \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \boxed{\ln|x| - \ln|x-1| - \frac{1}{x-1} + C}$$

Question 3:

(a)[5] A sphere of radius r has volume $V = 4\pi r^3/3$. Derive this formula using integration.



$y = \sqrt{r^2 - x^2}$: rotate about x-axis.

$A(x) = \pi\left[\sqrt{r^2-x^2}\right]^2 = \pi(r^2 - x^2)$

$$\therefore V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

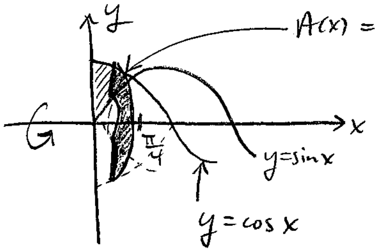
$$= 2\pi \int_0^r r^2 - x^2\,dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi\left[r^3 - \frac{r^3}{3}\right]$$

$$= \boxed{\frac{4}{3}\pi r^3}$$

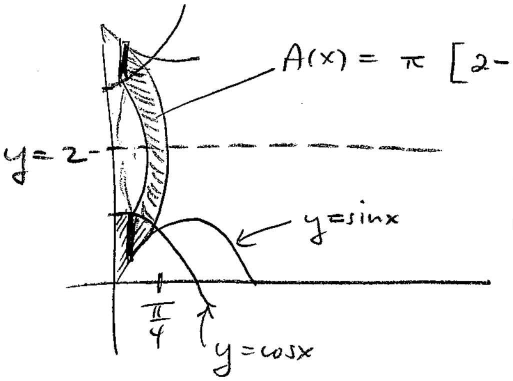
(b)[2] The region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$ is rotated about the line x-axis. Set up but do not evaluate the integral representing the volume of the resulting solid.



$A(x) = \pi\cos^2 x - \pi\sin^2 x$

$$\boxed{\therefore V = \pi \int_0^{\frac{\pi}{4}} \cos^2 x - \sin^2 x \,dx}$$

(b)[3] The region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$ is rotated about the line $y = 2$. Set up but do not evaluate the integral representing the volume of the resulting solid.



$A(x) = \pi\left[2 - \sin x\right]^2 - \pi\left[2 - \cos x\right]^2$

$$\boxed{\therefore V = \pi \int_0^{\frac{\pi}{4}} [2-\sin x]^2 - [2-\cos x]^2\,dx}$$

Question 4:

(a)[5] Evaluate the following improper integral. Show all steps including any required limits. Note that p is constant and $p > 1$.

$$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

$$I = \int \frac{1}{x(\ln x)^p}\,dx \quad \left.\right\} \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}$$

$$= \int u^{-p}\,du$$

$$= \frac{u^{1-p}}{1-p}$$

$$= \frac{(\ln x)^{1-p}}{1-p} \quad \left.\right\} \text{ note: } 1-p > 0$$

$$\therefore \int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^p}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{(\ln x)^{1-p}}{1-p}\right]_e^t$$

$$= \lim_{t\to\infty} \left[\frac{(\ln t)^{1-p}}{1-p} - \frac{1}{1-p}\right]$$

$$= \boxed{\frac{1}{p-1}}$$

(b)[5] Solve the following differential equation:

$$(1+\cos x)y' = e^{-y}\sin x, \qquad y(0) = 0$$

You may leave your solution in implicit form (it is not necessary to isolate y in your final answer.)

$$(1+\cos x)\frac{dy}{dx} = e^{-y}\sin x$$

$$\int e^y\,dy = \int \frac{\sin x}{1+\cos x}\,dx$$

$$e^y = -\ln|1+\cos x| + C_1$$

$$y(0)=0 \Rightarrow \quad e^0 = -\ln|1+\cos 0| + C_1$$

$$\therefore C_1 = 1 + \ln 2.$$

$$\therefore \boxed{e^y = -\ln|1+\cos x| + 1 + \ln 2}$$

Question 5:

(a)[5 points] Use T_4, the trapezoid rule on four subintervals, to approximate the integral $\int_0^{\pi} \sin^2 x\, dx$.

$$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$\therefore \quad T_4 = \frac{\Delta x}{2}\left[f(0) + 2f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{3\pi}{4}\right) + f(\pi)\right]$$

where $f(x) = \sin^2 x$

$$= \frac{\left(\frac{\pi}{4}\right)}{2}\left[\cancel{\sin^2(0)}^{\,0} + 2\sin^2\left(\tfrac{\pi}{4}\right) + 2\sin^2\left(\tfrac{\pi}{2}\right) + 2\sin^2\left(\tfrac{3\pi}{4}\right) + \underbrace{\cancel{\sin^2(\pi)}}_{0}\right]$$

$$= \frac{\pi}{8}\left[(2)\left(\tfrac{1}{2}\right) + (2)(1) + (2)\left(\tfrac{1}{2}\right)\right]$$

$$= \boxed{\frac{\pi}{2}}$$

(b)[5 points] Determine the error in your approximation T_4 in part (a). Recall, the error in using the trapezoid rule to approximate $\int_a^b f(x)\, dx$ is at most $\frac{K(b-a)^3}{12n^2}$, where $|f''(x)| \leq K$ on $[a, b]$.

$$f(x) = \sin^2 x$$

$$f'(x) = 2\sin x \cos x = \sin(2x)$$

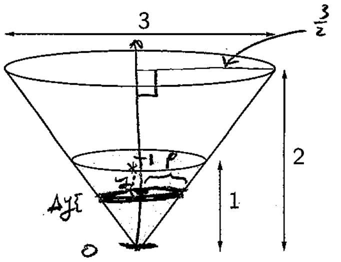
$$f''(x) = 2\cos(2x).$$

$\therefore \quad |f'(x)| = |2\cos(2x)| \leq 2$ on $[0, \pi]$, so take $K = 2$

$$\therefore \quad E_{T_4} \leq \frac{2(\pi - 0)^3}{12 \cdot 4^2} = \frac{2\pi^3}{(12)(16)} = \boxed{\frac{\pi^3}{96}}$$

Question 6:

(a)[5] The cone shaped vessel shown below has been filled with water to a depth of 1 m. Determine the amount of work required to empty the cone by pumping water up and over the top rim. Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2.



Slice at y has radius ρ.

By similar triangles, $\frac{\rho}{y_i^*} = \frac{\frac{3}{2}}{2}$

$\therefore \rho = \frac{3}{4} y_i^*$

$\therefore$ Weight of slice is

(volume)(density) g

$= \left[\pi \left(\frac{3}{4} y_i^*\right)^2 \Delta y\right] \rho g$

$\therefore$ Work to lift this slice to top of tank is (weight)(distance)

$= \left(\frac{9}{16} \pi (y_i^*)^2 \Delta y \rho g\right)(2 - y_i^*)$

$= \frac{9}{16} \pi \rho g (y_i^*)^2 (2 - y_i^*) \Delta y$

$\therefore$ Total work $\approx \sum_{i=1}^{n} \frac{9}{16} \pi \rho g (y_i^*)^2 (2 - y_i^*) \Delta y$

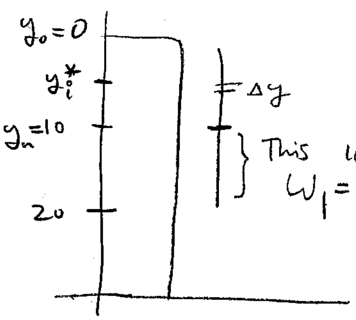
Letting $n \to \infty$:

$$W = \frac{9}{16} \pi \rho g \int_0^1 y^2 (2 - y)\, dy$$

$$= \frac{9}{16} \pi \rho g \left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^1 = \boxed{\frac{15}{64} \pi \rho g \ \text{J}}$$

(b)[5] A 20 m rope of mass 4 kg hangs over the side of a building. How much work is done in pulling half of the rope up onto the roof? Recall that acceleration due to gravity is $g = 9.8$ m/s^2.

linear density $= \frac{4 \text{ kg}}{20 \text{ m}} = \frac{1}{5} \frac{\text{kg}}{\text{m}}$



This 10 m segment lifted 10 m, so

$W_1 = (10)\left(\frac{1}{5}\right) g \cdot 10 = 20g$ J.

$$W_2 \approx \sum_{i=1}^{n} y_i \frac{1}{5} g \Delta y \to \frac{1}{5} g \int_0^{10} y\, dy \quad \text{as } n \to \infty$$

$$= \frac{1}{5} g \left[\frac{y^2}{2}\right]_0^{10}$$

$$= \frac{1}{5} g \cdot 50$$

$$= 10g \text{ J}.$$

$\therefore$ Total work is

$\boxed{W = W_1 + W_2 = 30g \text{ J}.}$