Question 1:

(a)[5] Determine $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}\,\sin(\sqrt{x})}\,dx = I$

Let $u = \sin(\sqrt{x})$, $du = \dfrac{\cos(\sqrt{x})}{2\sqrt{x}}\,dx$

$\therefore I = 2\displaystyle\int \frac{1}{u}\,du$

$= 2\ln|u| + C$

$= \boxed{2\ln|\sin(\sqrt{x})| + C}$

(b)[5] Determine $I = \displaystyle\int \sec^3 x\,\tan^3 x\,dx = \int \sec^2 x\,\tan^2 x\,\sec x\tan x\,dx$

$= \displaystyle\int \sec^2 x\,(\sec^2 x - 1)\sec x\tan x\,dx$

Let $u = \sec x$, $du = \sec x\tan x\,dx$

$\therefore I = \displaystyle\int u^2(u^2-1)\,du$

$= \displaystyle\int u^4 - u^2\,du$

$= \dfrac{u^5}{5} - \dfrac{u^3}{3} + C$

$= \boxed{\dfrac{\sec^5 x}{5} - \dfrac{\sec^3 x}{3} + C}$

Question 2:

(a)[5] Compute) $\int_0^{1/2} \arctan(2x)\,dx$

For $I = \int \arctan(2x)\,dx$: $u = \arctan(2x)$, $dv = dx$

$du = \frac{2}{1+4x^2}dx$, $v = x$

$\therefore I = x\arctan(2x) - \int \frac{2x}{1+4x^2}dx$ $\Big\}$ let $u = 1+4x^2$, $du = 8x\,dx$

$= x\arctan(2x) - \frac{1}{4}\ln|1+4x^2|$

$\therefore \int_0^{1/2} \arctan(2x)dx = \left[x\arctan(2x) - \frac{1}{4}\ln|1+4x^2|\right]_0^{1/2}$

$= \frac{1}{2}\arctan(1) - \frac{1}{4}\ln|1+4(\frac{1}{4})| - 0 + 0$

$= (\frac{1}{2})(\frac{\pi}{4}) - \frac{1}{4}\ln 2$

$= \boxed{\frac{\pi}{8} - \frac{1}{4}\ln 2}$

(b)[5] Determine $I = \int \frac{1}{x(x-1)^2}dx$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + B(x)(x-1) + C(x)}{x(x-1)^2}$$

$$= \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2}$$

$\therefore A = 1$; $A+B=0 \Rightarrow B=-1$

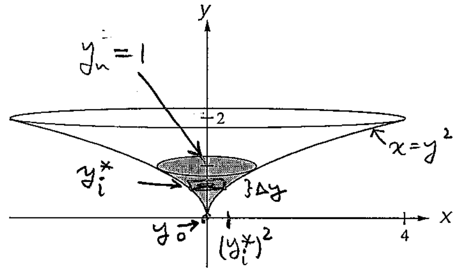
$-2A-B+C=0 \Rightarrow C = 2A+B = 1$

$$\therefore I = \int \frac{1}{x} - \frac{1}{x-1} + (x-1)^{-2}\,dx$$

$$= \boxed{\ln|x| - \ln|x-1| - \frac{1}{x-1} + C}$$

Question 3:

(a)[5 points] A vessel is formed by rotating the curve $y = \sqrt{x}$ about the y-axis as shown below. The vessel has a top radius of 4 m and a depth of 2 m. If the vessel is initially filled with water to a depth of 1 m, how much work is required to empty the vessel by pumping the water up and over the top rim? Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2.



Disc of water at y_i^* has weight

$$F_i = (\text{volume})\rho g$$

$$= \pi[(y_i^*)^2]^2 \rho g \Delta y \text{ N}$$

$\therefore$ Work required to lift this disc to top of vessel is

$$W_i = (\text{force})(\text{distance})$$

$$= \pi(y_i^*)^4 \rho g (2-y_i^*)\Delta y \text{ Nm}$$

$$\therefore \text{Total work} = \lim_{n\to\infty} \sum_{i=1}^{n} \pi (y_i^*)^4 \rho g (2-y_i^*)\Delta y$$

$$= \int_0^1 \pi \rho g\, y^4(2-y)\,dy$$

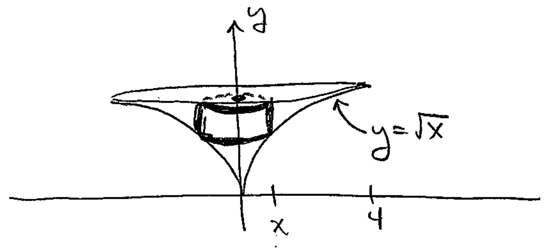
$$= \pi\rho g\left[\frac{2y^5}{5} - \frac{y^6}{6}\right]_0^1$$

$$= \pi\rho g\left[\frac{2}{5} - \frac{1}{6}\right]$$

$$= \boxed{\frac{7\pi\rho g}{30}} \approx \boxed{7184 \text{ J}}$$

(b)[5 points] Determine the total volume of the vessel described in part (a).

Using cylindrical shells:



$$V = \int_0^4 2\pi\, x(2-\sqrt{x})\,dx$$

$$= 2\pi \int_0^4 2x - x^{3/2}\,dx$$

$$= 2\pi\left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$$

$$= 2\pi\left[16 - \frac{64}{5}\right] = \boxed{\frac{32\pi}{5}}$$

Question 4:

(a)[5] Determine the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1+(y')^2}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\frac{\pi}{3}}$$

$$= \ln|2+\sqrt{3}| - \cancel{\ln|1+0|} \to 0$$

$$= \boxed{\ln(2+\sqrt{3})}$$

(b)[5] Solve the following differential equation:

$$(1+\cos x)y' = e^{-y}\sin x, \qquad y(0) = 0$$

You may leave your solution in implicit form (it is not necessary to isolate y in your final answer.)

$$\int e^y\, dy = \int \frac{\sin x}{1+\cos x}\, dx \quad \Big\} \text{ let } u = 1+\cos x, \quad du = -\sin x\, dx$$

$$e^y = -\ln|1+\cos x| + C$$

$\underline{y(0) = 0}$:

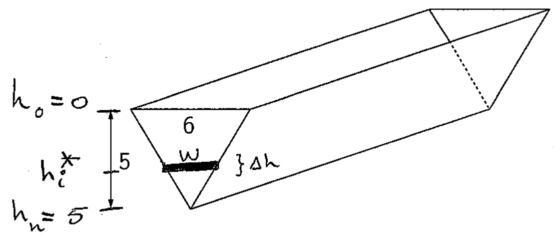
$$e^0 = -\ln|1+\cos(0)| + C$$

$$\therefore\ 1 + \ln(2) = C$$

$$\therefore\ \boxed{e^y = -\ln|1+\cos x| + (1+\ln 2).}$$

Question 5:

(a)[5 points] A large 5 m deep trough has triangular ends of top width 6 m. If the trough is full of water determine the hydrostatic force (force due to water pressure) exerted on one end of the trough. Recall that the density of water is $\rho = 1000$ kg/m^3, acceleration due to gravity is $g = 9.8$ m/s^2, and pressure P as a function of depth h is $P(h) = \rho g h$.



By similar Δ's:

$$\frac{w}{6} = \frac{5-h_i^*}{5}$$

$$\therefore w = \frac{6}{5}(5-h_i^*).$$

area of strip at h_i^* $= w\,\Delta h = \frac{6}{5}(5-h_i^*)\Delta h$

$\therefore$ pressure on strip $= [\rho g h_i^*]\left[\frac{6}{5}(5-h_i^*)\Delta h\right]$

$\therefore$ Total pressure $= \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho g h_i^* \frac{6}{5}(5-h_i^*)\Delta h$

$$= \int_0^5 \rho g h \cdot \frac{6}{5}(5-h)\,dh$$

$$= \frac{6}{5}\rho g \int_0^5 5h - h^2\,dh$$

$$= \frac{6}{5}\rho g \left[\frac{5h^2}{2} - \frac{h^3}{3}\right]_0^5$$

$$= \frac{6}{5}\rho g\left[\frac{125}{2} - \frac{125}{3}\right] = 25\rho g \approx \boxed{245\,000 \text{ N}}$$

(b)[5 points] Determine the first three nonzero terms of the Maclaurin series for $f(x) = x\sin(x^4)$.

$$x\sin(x^4)$$

$$= x\left[(x^4) - \frac{(x^4)^3}{3!} + \frac{(x^4)^5}{5!} - \cdots\right]$$

$$= x^5 - \frac{x^{13}}{3!} + \frac{x^{21}}{5!} - \cdots$$

Question 6:

(a)[5 points] Use a Maclaurin series (not L'Hospital's Rule) to evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$$

$$= \lim_{x\to 0} \frac{1-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right]}{1+x-\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right]}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\cdots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\cdots} \quad \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2!}-\frac{x^2}{4!}+\cdots}{-\frac{1}{2!}-\frac{x}{3!}-\cdots} = \boxed{-1}$$

(b)[2 points] Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$ converges or diverges.

$$\frac{n^2+1}{n^3+1} \geqslant \frac{n^2}{2n^3} = \frac{1}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series, $p=1$)

by the comparison test so does

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}.$$

(c)[3 points] Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ converges or diverges.

$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\,dx = \lim_{t\to\infty} \ln\left[\ln(x+1)\right]\Big|_1^t$$

$$= \lim_{t\to\infty} \ln\left[\ln(t+1)\right] - \ln\left[\ln(2)\right]$$

$$= \infty.$$

$\therefore$ By the integral test $\sum_{n=1} \frac{1}{(n+1)\ln(n+1)}$ $\boxed{\text{diverges.}}$