Question 1 [10]:

Determine $\int \frac{1}{x\sqrt{25-x^2}}\,dx$. (State your final answer without using inverse hyperbolic functions.)

Let $x = 5\sin\theta$

$dx = 5\cos\theta\, d\theta$

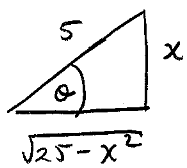
$$\therefore\ I = \int \frac{1}{x\sqrt{25-x^2}}\,dx$$

$$= \int \frac{\cancel{5}\cos\theta\, d\theta}{\cancel{5}\sin\theta\sqrt{25-25\sin^2\theta}}$$

$$= \frac{1}{5}\int \frac{1}{\sin\theta}\,d\theta$$

$$= \frac{1}{5}\int \csc\theta\, d\theta$$

$$= \frac{1}{5}\ln\left|\csc\theta - \cot\theta\right| + C$$



$$\therefore\ I = \frac{1}{5}\ln\left|\frac{5}{x} - \frac{\sqrt{25-x^2}}{x}\right| + C.$$

Question 2:

(a)[4] Determine $\int \sin^3 x \cos^7 x \, dx$

$$\int \sin^3 x \cos^7 x \, dx$$

$$= \int (1-\cos^2 x)\cos^7 x \sin x \, dx \qquad \text{let } u = \cos x, \quad du = -\sin x\, dx$$

$$= -\int (1-u^2)u^7 \, du$$

$$= \int u^9 - u^7 \, du$$

$$= \frac{u^{10}}{10} - \frac{u^8}{8} + C$$

$$= \boxed{\frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} + C}$$

(b)[6] Evaluate $\int_0^1 \arctan x \, dx$

For $I = \int \arctan x \, dx$ let $u = \arctan x$, $dv = dx$

$du = \frac{1}{1+x^2}dx$, $v = x$

$$\therefore I = \int u\,dv = uv - \int v\,du$$

$$= x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \qquad \text{let } u = 1+x^2, \quad du = 2x\,dx$$

$$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

$$\therefore \int_0^1 \arctan x \, dx = \left[x\arctan x - \frac{1}{2}\ln|1+x^2|\right]_0^1$$

$$= \arctan(1) - \frac{1}{2}\ln(2)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$$

Question 3 [10]:

Determine $\int \frac{3x^2+4}{x^3+2x}\,dx$.

$$\frac{3x^2+4}{x^3+2x} = \frac{3x^2+4}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$= \frac{Ax^2+2A+Bx^2+Cx}{x(x^2+2)}$$

$$= \frac{(A+B)x^2+Cx+2A}{x(x^2+2)}$$

$\therefore \quad C=0$

$2A=4 \Rightarrow A=2$

$A+B=3 \Rightarrow 2+B=3 \Rightarrow B=1$

$$\therefore \int \frac{3x^2+4}{x^3+2x}\,dx = \int \frac{2}{x}\,dx + \left(\frac{1}{2}\right)\underbrace{\int \frac{2x}{x^2+2}\,dx}_{\substack{u=x^2+2 \\ du=2x\,dx}}$$

$$\boxed{= 2\ln|x| + \frac{1}{2}\ln|x^2+2| + C}$$

Question 4:

(a)[5 points] Evaluate the improper integral $\int_3^5 \frac{1}{\sqrt{5-x}}\,dx$. Show all steps including any required limits.

$$\int_3^5 \frac{1}{\sqrt{5-x}}\,dx = \lim_{t\to 5^-} \int_3^t \frac{1}{\sqrt{5-x}}\,dx$$

$$= \lim_{t\to 5^-} \int_3^t (5-x)^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 5^-} \left[-2(5-x)^{\frac{1}{2}}\right]_3^t$$

$$= \lim_{t\to 5^-} \left[\underbrace{-2(5-t)^{\frac{1}{2}}}_{\to 0} + 2(5-3)^{\frac{1}{2}}\right]$$

$$= \boxed{2\sqrt{2}}$$

(b)[5 points] Determine if $\int_1^\infty \frac{\sin^2 x}{x^2+\sqrt{x}}\,dx$ converges or diverges. State reasons for your conclusion.

$$0 \leq \frac{\sin^2 x}{x^2+\sqrt{x}} \leq \frac{1}{x^2} \quad \text{on } [1,\infty).$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral, $p>1$),

by the comparison theorem,

$$\int_1^\infty \frac{\sin^2 x}{x^2+\sqrt{x}}\,dx \quad \text{converges.}$$

Question 5:

(a)[5 points] Use S_4, Simpson's rule on four subintervals, to approximate $\int_0^4 \sqrt{3+\cos^2(\pi x)}\,dx$.

$$\Delta x = \frac{4-0}{4} = 1$$

$$f(x) = \sqrt{3+\cos^2(\pi x)} \ ; \ f(k) = 2 \text{ for } k = 0,1,2,3,4$$

$$\therefore \int_0^4 f(x)\,dx \approx \frac{\Delta x}{3}\left[f(0)+4f(1)+2f(2)+4f(3)+f(4)\right]$$

$$= \frac{1}{3}\left[2+(4)(2)+(2)(2)+(4)(2)+2\right]$$

$$= \boxed{8}$$

(b)[5 points] The fourth derivative of $f(x) = \sqrt{3+\cos^2(\pi x)}$ is between -300 and 160 for every x. If we wish to approximate $\int_0^4 \sqrt{3+\cos^2(\pi x)}\,dx$ with accuracy 0.001 using Simpson's rule, how many subintervals are required?

$$E_{S_n} \leq \frac{K(b-a)^5}{180n^4} \quad \text{where } \left|f^{(iv)}(x)\right| \leq K \text{ on } [a,b].$$

Here $\left|f^{(iv)}(x)\right| \leq 300$, so we require

$$\frac{300(4-0)^5}{180n^4} \leq 0.001$$

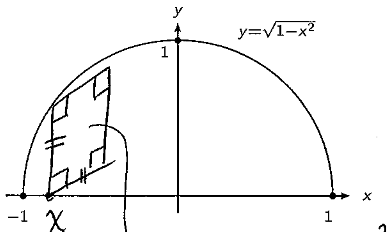
$$\Longrightarrow \quad n \geq \left[\frac{300(4)^5}{(180)(0.001)}\right]^{\frac{1}{4}}$$

$$\Longrightarrow \quad n \geq 36.1$$

$$\Longrightarrow \quad \boxed{n = 38 \text{ since } n \text{ must be even}}$$

Question 6:

(a)[5 points] The base of a solid is the region bounded between the curve $y = \sqrt{1-x^2}$ and the x axis. Cross-sections perpendicular to both the base and the x-axis are squares. Find the volume of the solid.



$$A(x) = \left(\sqrt{1-x^2}\right)^2 = 1-x^2$$

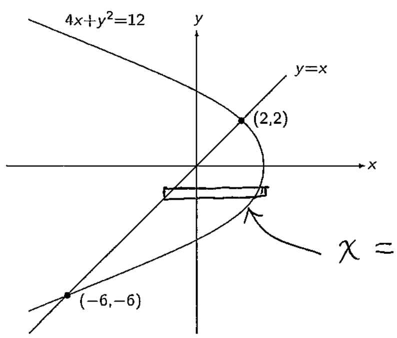
$$\therefore V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 1-x^2\,dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[1-\frac{1}{3}\right]$$

$$= \boxed{\frac{4}{3}}$$

(b)[5 points] Find the area of the region bounded by the curves $4x + y^2 = 12$ and $y = x$.



$$x = 3 - \frac{1}{4}y^2$$

$$A = \int_{y=-6}^{y=2} \left(3 - \frac{1}{4}y^2 - y\right) dy$$

$$= \left[3y - \frac{1}{12}y^3 - \frac{y^2}{2}\right]_{-6}^{2}$$

$$= \left(6 - \frac{2}{3} - 2\right) - \left(\cancel{-18} \cancel{+18} - 18\right)$$

$$= \boxed{\frac{64}{3}}$$