Question 1:

(a)[6] Evaluate $\int_0^1 \arctan x \, dx$

For $I = \int \arctan x \, dx$, let $u = \arctan x$, $dv = dx$

$du = \frac{1}{1+x^2} dx$, $v = x$.

$$\therefore I = \int u \, dv = uv - \int v \, du = x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \arctan x - \frac{1}{2} \ln|1+x^2| + C.$$

$$\therefore \int_0^1 \arctan x \, dx = \left[x \arctan x - \frac{1}{2} \ln|1+x^2| \right]_0^1$$

$$= \arctan(1) - \frac{1}{2} \ln(2)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2} \ln(2)}$$

(b)[4] Determine $\int \sin^3 x \cos^7 x \, dx$

$$\int \sin^3 x \cos^7 x \, dx$$

$$= \int (1 - \cos^2 x) \cos^7 x \sin x \, dx$$

let $u = \cos x$

$du = -\sin x \, dx$

$$= -\int (1-u^2) u^7 \, du$$

$$= \int u^9 - u^7 \, du$$

$$= \frac{u^{10}}{10} - \frac{u^8}{8} + C$$

$$= \boxed{\frac{\cos^{10} x}{10} - \frac{\cos^8 x}{8} + C}$$

Question 2 [10]:

Determine $\displaystyle\int \frac{1}{x\sqrt{9-x^2}}\,dx$. (State your final answer without using inverse hyperbolic functions.)

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

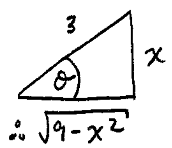
$$\therefore \quad I = \int \frac{1}{x\sqrt{9-x^2}}\,dx$$

$$= \int \frac{\cancel{3}\cos\theta\, d\theta}{\cancel{3}\sin\theta\sqrt{9-9\sin^2\theta}}$$

$$= \frac{1}{3}\int \frac{1}{\sin\theta}\,d\theta$$

$$= \frac{1}{3}\int \csc\theta\, d\theta$$

$$= \frac{1}{3}\ln\left|\csc\theta - \cot\theta\right| + C.$$



$$\boxed{\therefore \quad I = \frac{1}{3}\ln\left|\frac{3}{x} - \frac{\sqrt{9-x^2}}{x}\right| + C.}$$

Question 3 [10]:

Determine $\int \frac{3x^2+8}{x^3+4x}\,dx$.

$$I = \int \frac{3x^2+8}{x(x^2+4)}\,dx.$$

$$\frac{3x^2+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$= \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}.$$

$\therefore$ $C=0$

$4A=8 \Rightarrow A=2$

$A+B=3 \Rightarrow B=1$

$$\therefore \quad I = \int \frac{2}{x} + \frac{x}{x^2+4}\,dx$$

$$= \int \frac{2}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx$$

$$= \boxed{2\ln|x| + \frac{1}{2}\ln|x^2+4| + C.}$$

Question 4:

(a)[5 points] Use S_4, Simpson's rule on four subintervals, to approximate $\int_0^4 \sqrt{1+\sin^2(\pi x)}\,dx$.

$$\Delta x = \frac{4-0}{4} = 1$$

$f(x) = \sqrt{1+\sin^2(\pi x)}$; $f(k) = 1$ for $k = 0, 1, 2, 3, 4$

$$\therefore \int_0^4 f(x)\,dx \approx \frac{\Delta x}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$$= \frac{1}{3}\left[1 + 4 + 2 + 4 + 1\right]$$

$$= \boxed{4}$$

(b)[5 points] The fourth derivative of $f(x) = \sqrt{1+\sin^2(\pi x)}$ is between -700 and 200 for every x. If we wish to approximate $\int_0^4 \sqrt{1+\sin^2(2\pi)}\,dx$ with accuracy 0.001 using Simpson's rule, how many subintervals are required?

$E_{S_n} \leq \dfrac{K(b-a)^5}{180n^4}$ where $\left|f^{(iv)}(x)\right| \leq K$ on $[a,b]$.

Here $\left|f^{(iv)}(x)\right| \leq 700$, so we require

$$\frac{700(4-0)^5}{180n^4} \leq 0.001$$

$$\Rightarrow \quad n \geq \left[\frac{700(4)^5}{180(0.001)}\right]^{\frac{1}{4}}$$

$$\Rightarrow \quad n \geq 44.7$$

$\therefore$ $n = 46$ since n must be even

Question 5:

(a)[5 points] Evaluate the improper integral $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$.

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} \int_2^t (3-x)^{-\frac{1}{2}}\,dx$$

$$= \lim_{t\to 3^-} \Big[-2(3-x)^{\frac{1}{2}}\Big]_2^t$$

$$= \lim_{t\to 3^-} \Big[\underbrace{-2(3-t)^{\frac{1}{2}}}_{\to 0} + 2\,(3-2)^{\frac{1}{2}}\Big]$$

$$= \boxed{2}$$

(b)[5 points] Determine if $\int_1^\infty \frac{\sin^2 x}{x^2+\sqrt{x}}\,dx$ converges or diverges. (Do not attempt to evaluate the integral).

$$0 \le \frac{\sin^2 x}{x^2+\sqrt{x}} \le \frac{1}{x^2} \quad \text{on} \quad [1,\infty).$$

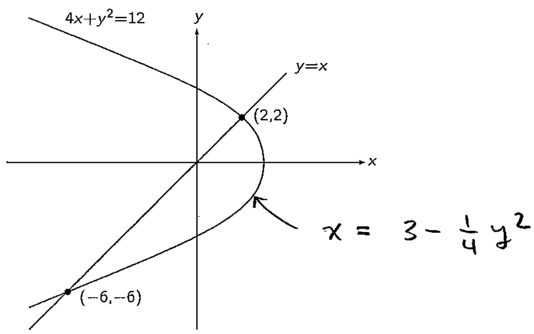
Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral, $p>1$)

by the comparison theorem,

$\int_1^\infty \frac{\sin^2 x}{x^2+\sqrt{x}}\,dx$ Converges.

Question 6:

(a)[5 points] Find the area of the region bounded by the curves $4x + y^2 = 12$ and $y = x$.



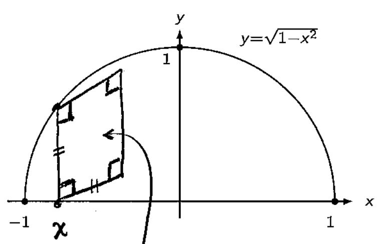
$$\therefore A = \int_{y=-6}^{y=2} \left(3 - \frac{1}{4}y^2 - y\right) dy$$

$$= \left[3y - \frac{1}{12}y^3 - \frac{y^2}{2}\right]_{-6}^{2}$$

$$= \left(6 - \frac{2}{3} - 2\right) - \left(-18 + 18 - 18\right)$$

$$= \boxed{\frac{64}{3}}$$

(b)[5 points] The base of a solid is the region bounded between the curve $y = \sqrt{1 - x^2}$ and the x axis. Cross-sections perpendicular to both the base and the x-axis are squares. Find the volume of the solid.



$$A(x) = \left(\sqrt{1 - x^2}\right)^2 = 1 - x^2$$

$$\therefore V = \int_{-1}^{1} A(x)\, dx = 2\int_0^1 1 - x^2\, dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[1 - \frac{1}{3}\right]$$

$$= \boxed{\frac{4}{3}}$$