Question 1:

(a)[3] Determine the average value of $f(x) = \dfrac{\sin x}{2 - \cos x}$ over the interval $[0, \pi]$.

$$f_{ave} = \frac{1}{\pi - 0} \int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx \quad \left\{ \begin{array}{l} \text{let } u = 2-\cos x \\ du = \sin x\,dx \end{array} \right. \left\{ \begin{array}{l} x=0 \Rightarrow u=1 \\ x=\pi \Rightarrow u=3 \end{array} \right.$$

$$= \frac{1}{\pi} \int_1^3 \frac{1}{u}\,du$$

$$= \frac{1}{\pi} \Big[\ln|u|\Big]_1^3$$

$$= \frac{1}{\pi}\left(\ln|3| - \cancelto{0}{\ln|1|}\right)$$

$$= \boxed{\frac{\ln(3)}{\pi}}$$

(b)[3] Without evaluating the integral, give upper and lower estimates of $\int_0^2 \sqrt{1 + e^{(x/2)}}\,dx$.

$$\sqrt{1+e^{0/2}} \leq \sqrt{1+e^{(x/2)}} \leq \sqrt{1+e^{2/2}} \quad \text{on } [0,2].$$

$$\Rightarrow \quad \sqrt{2} \leq \sqrt{1+e^{(x/2)}} \leq \sqrt{1+e}$$

$$\Rightarrow \quad \sqrt{2}\,(2-0) \leq \int_0^2 \sqrt{1+e^{(x/2)}}\,dx \leq \sqrt{1+e}\,(2-0)$$

$$\Rightarrow \quad 2\sqrt{2} \leq \int_0^2 \sqrt{1+e^{x/2}}\,dx \leq 2\sqrt{1+e}$$

(c)[4] Determine the constant a and the function $f(x)$ if

$$a \int_a^x f(t)\,dt = \sqrt{x-3}$$

Letting $x=a$ give $a \cancelto{0}{\int_a^a f(t)\,dt} = \sqrt{a-3}$ so $\boxed{a=3}$

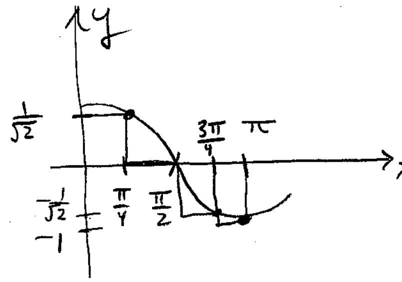
Now, using FTC I: $\dfrac{d}{dx}\left[3\int_3^x f(t)\,dt\right] = \dfrac{d}{dx}\left[\sqrt{x-3}\right]$

$$3\,f(x) = \frac{1}{2\sqrt{x-3}}$$

$$\therefore \boxed{f(x) = \frac{1}{6\sqrt{x-3}}}$$

Question 2:

(a)[3] Estimate $\int_0^{\pi} \cos x\, dx$ using four subintervals and right endpoints.



$\Delta x = \frac{\pi}{4}$

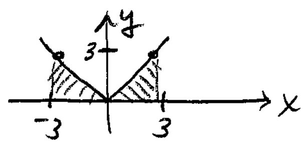
$$\int_0^{\pi} \cos(x)\,dx \approx R_4 = \frac{\pi}{4}\left[\cancel{\frac{1}{\sqrt{2}}} + 0 + \cancel{\frac{-1}{\sqrt{2}}} + (-1)\right]$$

$$= \boxed{-\frac{\pi}{4}}$$

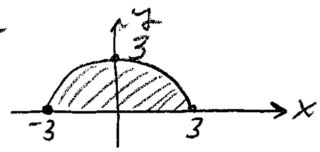
(b)[4] Use an area interpretation to evaluate $\int_{-3}^{3} \left(|x| + \sqrt{9-x^2}\right) dx$.

$$I = \int_{-3}^{3} |x| + \sqrt{9-x^2}\,dx = \int_{-3}^{3} |x|\,dx + \int_{-3}^{3} \sqrt{9-x^2}\,dx$$

- $y = |x|$ has graph



- $y = \sqrt{9-x^2}$ has graph



$$\therefore I = (2)\left(\tfrac{1}{2}\right)(3)(3) + \tfrac{1}{2}\pi(3)^2 = \boxed{9 + 9\frac{\pi}{2}}$$

(c)[3] Determine the value of

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 - \left(i\frac{3}{n}\right)^2\right] \frac{3}{n}$$

by writing it as the definite integral of some function $f(x)$ over an interval $[a, b]$.

Here $\Delta x = \frac{3}{n}$, $x_i = 0 + i\left(\frac{3}{n}\right)$, $f(x_i) = 1 - x_i^2$, $[a,b] = [0,3]$

$$\therefore I = \int_0^3 1 - x^2\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^3$$

$$= \left(3 - \frac{3^3}{3}\right) - \left(0 - \frac{0^3}{3}\right)$$

$$= \boxed{-6}$$

Question 3:

(a)[3] Evaluate $\int_0^1 (1-x)^{13}\,dx$.

Let $u = 1-x$, $du = -dx$ $\left.\begin{matrix} x=0 \Rightarrow u=1 \\ x=1 \Rightarrow u=0 \end{matrix}\right\}$

$$\therefore \int_0^1 (1-x)^{13}dx = -\int_1^0 u^{13}du$$
$$= \int_0^1 u^{13}du$$
$$= \left[\frac{u^{14}}{14}\right]_0^1$$
$$= \boxed{\frac{1}{14}}$$

(b)[3] Evaluate $\frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+4x}}\,dx$.

Let $u = x^2+4x$
$du = 2x+4$

$$\therefore \int \frac{x+2}{\sqrt{x^2+4x}}du = \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$
$$= \frac{1}{2}\int u^{-\frac{1}{2}}du$$
$$= \cancel{\tfrac{1}{2}}\,\cancel{2}\,u^{1/2} + C$$
$$= \sqrt{x^2+4x} + C$$

(c)[4] Evaluate $\int \cos(x)\sin(\sin(x))\,dx$.

Let $u = \sin(x)$
$du = \cos(x)\,dx$

$$\therefore \int \cos(x)\sin(\sin(x))dx = \int \sin(u)du$$
$$= -\cos(u) + C$$
$$= \boxed{-\cos(\sin(x)) + C}$$

Question 4:

(a)[3] Evaluate $\int \frac{1}{2x\sqrt{\ln x}}\,dx$.

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$\therefore \int \frac{1}{2x\sqrt{\ln x}}\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\, 2\, u^{\frac{1}{2}} + C$$

$$= \boxed{\sqrt{\ln x} + C}$$

(b)[4] Evaluate $\int_0^a x\sqrt{a^2 - x^2}\,dx$ where $a > 0$ is a constant.

Let $u = a^2 - x^2$; $x = 0 \Rightarrow u = a^2$

$du = -2x\,dx$; $x = a \Rightarrow u = 0$

$$\therefore -\frac{1}{2}\int_0^a (-2)\, x\sqrt{a^2 - x^2}\,dx = -\frac{1}{2}\int_{a^2}^{0} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\int_0^{a^2} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\left[u^{3/2}\right]_0^{a^2}$$

$$= \boxed{\frac{a^3}{3}}$$

(c)[3] Evaluate $\int \frac{1 - e^t}{e^t}\,dt$.

$$\int \frac{1 - e^t}{e^t}\,dt = \int e^{-t} - 1\,dt$$

$$= \underbrace{\int e^{-t}\,dt}_{\text{let } u = -t,\ du = -dt} - t$$

$$= -\int e^u\,du - t$$

$$= -e^u - t + C$$

$$= \boxed{-e^{-t} - t + C}$$

Question 5:

(a)[3] Evaluate each of the following:

(i) $\frac{d}{dx}\underbrace{\int_0^1 e^{\arcsin x}\,dx}_{\text{constant}} = \boxed{0}$

(ii) $\frac{d}{dx}\int_0^x e^{\arcsin t}\,dt = \boxed{e^{\arcsin(x)}}$ by FTC I

(iii) $\int_0^1 \frac{d}{dx}(e^{\arcsin x})\,dx = \left[e^{\arcsin(x)}\right]_0^1 = e^{\arcsin(1)} - e^{\arcsin(0)}$

$= e^{\pi/2} - e^0$

$= \boxed{e^{\frac{\pi}{2}} - 1}$

(b)[4] A town's population is currently 10,000 and is growing at a rate of $r(t) = 200(1+\frac{t}{25})$ people per year, where t is in years and $t=0$ corresponds to the present. What will the town population be in two years time?

Pop. at time t years $= 10{,}000 + \int_0^2 200\left(1+\frac{t}{25}\right)dt$

$= 10{,}000 + 200\left[t + \frac{1}{25}\frac{t^2}{2}\right]_0^2$

$= 10{,}000 + 200\left[2 + \frac{2}{25}\right]$

$= \boxed{10{,}416 \text{ people}}$

(c)[3] Suppose $\int_0^{12} f(x)\,dx = 5$. Determine $\int_0^3 f(4x)\,dx$.

let $u = 4x$, $du = 4dx$ $\left\{\begin{array}{l} x=0 \Rightarrow u=0 \\ x \Rightarrow 3 \Rightarrow u = 12 \end{array}\right.$

$\therefore \int_0^3 f(4x)\,dx = \frac{1}{4}\int_0^3 f(4x)\,4dx$

$= \frac{1}{4}\int_0^{12} f(u)\,du$

$= \boxed{\frac{5}{4}}$