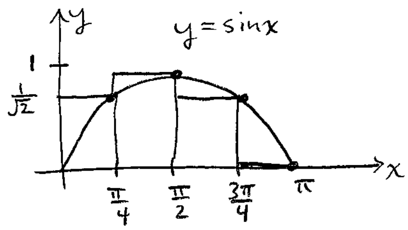
Question 1:

(a)[3] Estimate $\int_0^{\pi} \sin x \, dx$ using four subintervals and right endpoints.



$\Delta x = \frac{\pi}{4}$

$$\int_0^{\pi} \sin x \, dx \approx R_4 = \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0\right]$$
$$= \boxed{\frac{(1+\sqrt{2})\pi}{4}}.$$

(b)[3] Determine the value of

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \left(i\frac{3}{n}\right)^2\right]\frac{3}{n}$$

by writing it as the definite integral of some function $f(x)$ over an interval $[a, b]$.

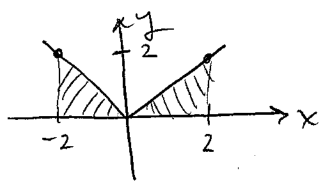
$\Delta x = \frac{3}{n}$, $x_i = 0 + i\,\frac{3}{n}$, $f(x_i) = 1 + x_i^2$, $[a,b] = [0,3]$

$$\therefore\ I = \int_0^3 1 + x^2 \, dx$$
$$= \left[x + \frac{x^3}{3}\right]_0^3$$
$$= (3+9) - (0+0)$$
$$= \boxed{12}$$

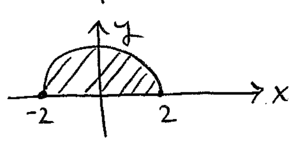
(c)[4] Use an area interpretation to evaluate $\int_{-2}^{2} \left(|x| + \sqrt{4-x^2}\right) dx$.

$$I = \int_{-2}^{2} |x| + \sqrt{4-x^2}\, dx = \int_{-2}^{2} |x| \, dx + \int_{-2}^{2} \sqrt{4-x^2}\, dx.$$

- $y = |x|$ has graph



- $y = \sqrt{4-x^2}$ has graph



$$\therefore\ I = (2)\left(\tfrac{1}{2}\right)(2)(2) + \tfrac{1}{2}\pi(2^2) = \boxed{4 + 2\pi}$$

Question 2:

(a)[3] Without evaluating the integral, give upper and lower estimates of $\int_0^2 \sqrt{1+e^{(x/2)}}\,dx$.

$$\sqrt{1+e^{\frac{0}{2}}} \leq \sqrt{1+e^{\frac{x}{2}}} \leq \sqrt{1+e^{\frac{2}{2}}} \quad \text{on } [0,2]$$

$$\Rightarrow \quad \sqrt{2} \leq \sqrt{1+e^{x/2}} \leq \sqrt{1+e}$$

$$\Rightarrow \quad \sqrt{2}\,(2-0) \leq \int_0^2 \sqrt{1+e^{x/2}}\,dx \leq \sqrt{1+e}\,(2-0)$$

$$2\sqrt{2} \leq \int_0^2 \sqrt{1+e^{x/2}}\,dx \leq 2\sqrt{1+e}$$

(b)[3] Determine the average value of $f(x) = \dfrac{\sin x}{2-\cos x}$ over the interval $[0,\pi]$.

$$f_{ave} = \frac{1}{\pi-0}\int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx \quad \left.\begin{matrix} \text{let } u = 2-\cos x \\ du = \sin x\,dx \end{matrix}\right\} \left\{\begin{matrix} x=0 \Rightarrow u=1 \\ x=\pi \Rightarrow u=3 \end{matrix}\right.$$

$$= \frac{1}{\pi}\int_1^3 \frac{1}{u}\,du$$

$$= \frac{1}{\pi}\Big[\ln|u|\Big]_1^3$$

$$= \frac{1}{\pi}\Big(\ln|3| - \ln|1|\Big) = \boxed{\frac{\ln(3)}{\pi}}$$

(c)[4] Determine the constant a and the function $f(x)$ if

$$a\int_a^x f(t)\,dt = \sqrt{x-2}$$

Letting $x=a$ gives $\cancel{a\int_a^a f(t)\,dt}^{\to 0} = \sqrt{a-2}$ so $\boxed{a=2}$.

Now using FTC1:
$$\frac{d}{dx}\left[2\int_2^x f(t)\,dt\right] = \frac{d}{dx}\left[\sqrt{x-2}\right]$$

$$2\,f(x) = \frac{1}{2\sqrt{x-2}}$$

$$\boxed{\therefore\ f(x) = \frac{1}{4\sqrt{x-2}}}$$

Question 3:

(a)[3] Evaluate $\int_0^1 (1-x)^{11}\,dx$.

Let $u = 1-x$

$du = -dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=0$

$$\therefore \int_0^1 (1-x)^{11}\,dx = -\int_1^0 u^{11}\,du$$
$$= \int_0^1 u^{11}\,du$$
$$= \left[\frac{u^{12}}{12}\right]_0^1$$
$$= \boxed{\frac{1}{12}}$$

(b)[4] Evaluate $\int \cos(x)\sin(\sin(x))\,dx$.

Let $u = \sin(x)$

$du = \cos(x)\,dx$

$$\therefore \int \cos(x)\sin\big(\sin(x)\big)\,dx$$
$$= \int \sin(u)\,du$$
$$= -\cos(u) + C$$
$$= \boxed{-\cos(\sin(x)) + C}$$

(c)[3] Evaluate $\frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+4x}}\,dx$.

Let $u = x^2+4x$

$du = 2x+4\,dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= \frac{1}{2}\int u^{-1/2}\,du$$
$$= \frac{1}{2}2u^{1/2} + C$$
$$= \boxed{\sqrt{x^2+4x} + C}$$

Question 4:

(a)[3] Evaluate $\int \frac{1+e^t}{e^t}\,dt$.

$$\int \frac{1+e^t}{e^t}\,dt = \int e^{-t}+1\,dt$$

$$= \underbrace{\int e^{-t}\,dt}_{\substack{u=-t \\ du=-dt}} + t$$

$$= -\int e^{u}\,du + t$$

$$= -e^{u} + t + C$$

$$= \boxed{-e^{-t}+t+C}$$

(b)[4] Evaluate $\int_0^a x\sqrt{a^2-x^2}\,dx$ where $a>0$ is a constant.

Let $u = a^2 - x^2$, $du = -2x\,dx$; $x=0 \Rightarrow u=a^2$, $x=a \Rightarrow u=0$

$$\therefore \left(-\tfrac{1}{2}\right)\int_0^a (-2)x\sqrt{a^2-x^2}\,dx = -\frac{1}{2}\int_{a^2}^0 u^{1/2}\,du$$

$$= \frac{1}{2}\int_0^{a^2} u^{1/2}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_0^{a^2}$$

$$= \boxed{\frac{a^3}{3}}$$

(c)[3] Evaluate $\int \frac{1}{2x\sqrt{\ln x}}\,dx$.

Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\therefore \int \frac{1}{2x\sqrt{\ln x}}\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{2}\,2\,u^{1/2} + C$$

$$= \sqrt{\ln x} + C$$

Question 5:

(a)[3] Evaluate each of the following:

(i) $\int_0^1 \frac{d}{dx}(e^{\arcsin x})\,dx$ $= \left[e^{\arcsin(x)}\right]_0^1 = e^{\arcsin(1)} - e^{\arcsin(0)}$

$= e^{\frac{\pi}{2}} - e^0$

$= \boxed{e^{\frac{\pi}{2}} - 1}$

(ii) $\frac{d}{dx}\underbrace{\int_0^1 e^{\arcsin x}\,dx}_{\text{constant}}$ $= \boxed{0}$

(iii) $\frac{d}{dx}\int_0^x e^{\arcsin t}\,dt$ $= \boxed{e^{\arcsin(x)}}$ by FTC 1

(b)[3] Suppose $\int_0^{12} f(x)\,dx = 5$. Determine $\underbrace{\int_0^4 f(3x)\,dx}$.

Let $u = 3x$, $du = 3\,dx$ $\left\{ \begin{array}{l} x=0 \Rightarrow u=0 \\ x=4 \Rightarrow u=12 \end{array}\right.$

$\therefore \frac{1}{3}\int_0^4 f(3x)\,3\,dx = \frac{1}{3}\int_0^{12} f(u)\,du$

$= \boxed{\frac{5}{3}}$

(c)[4] A town's population is currently 10,000 and is growing at a rate of $r(t) = 200(1 + \frac{t}{50})$ people per year, where t is in years and $t = 0$ corresponds to the present. What will the town population be in two years time?

pop. at time t years $= 10{,}000 + \int_0^2 200(1 + \frac{t}{50})\,dt$

$= 10{,}000 + 200\left[t + \frac{1}{50}\frac{t^2}{2}\right]_0^2$

$= 10{,}000 + 200\left[2 + \frac{2}{50}\right]$

$= \boxed{10{,}408 \text{ people.}}$