Question 1:

(a)[5] Determine $\int \frac{2e^{1/x}}{x^2}\,dx = I$

$u = \frac{1}{x}$, $du = \frac{-1}{x^2}\,dx$

$\therefore I = -2\int -\frac{e^{1/x}}{x^2}\,dx$

$= -2\int e^u\,du$

$= -2e^u + C$

$= \boxed{-2e^{1/x} + C}$

(b)[5] Determine $\int \sec^6 x \tan^2 x\,dx = I$

$I = \int \sec^4 x \tan^2 x \sec^2 x\,dx$

$= \int (\sec^2 x)^2 \tan^2 x \sec^2 x\,dx$

$= \int (1+\tan^2 x)^2 \tan^2 x \sec^2 x\,dx$

$u = \tan x$, $du = \sec^2 x\,dx$

$\therefore I = \int (1+u^2)^2 u^2\,du$

$= \int u^2 + 2u^4 + u^6\,du$

$= \frac{u^3}{3} + \frac{2}{5}u^5 + \frac{u^7}{7} + C$

$= \boxed{\frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C}$

Question 2:

(a)[5] Evaluate the improper integral (show all details) $\int_0^1 x \ln x \, dx = I_1$

$$I_1 = \lim_{t \to 0} \int_t^1 x \ln x \, dx .$$

For $I_2 = \int x \ln x$, integrate by parts: $u = \ln x \quad dv = x$

$du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$

$$\therefore I_2 = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \left(\frac{x^2}{2}\right) + C$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

$$\therefore I_1 = \lim_{t \to 0} \left[\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right]_t^1$$

$$= \lim_{t \to 0} \left[-\frac{1}{4} - \frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 \right]$$

$$= \boxed{-\frac{1}{4}}$$

Note: $\lim_{t \to 0} t^2 \ln t$

$$= \lim_{t \to 0} \frac{\ln t}{t^{-2}} \sim \frac{-\infty}{\infty}$$

$$\overset{H}{=} \lim_{t \to 0} \frac{(1/t)}{-2t^{-3}}$$

$$= \lim_{t \to 0} -\frac{1}{2} t^2 = 0$$

(L'Hospital's Rule)

(b)[5] Determine $\int \frac{x^2 + 3x + 2}{x(x^2 + 1)} dx = I$

$$\frac{x^2 + 3x + 2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}$$

$$\therefore A = 2, \quad C = 3, \quad A + B = 1 \Rightarrow B = -1$$

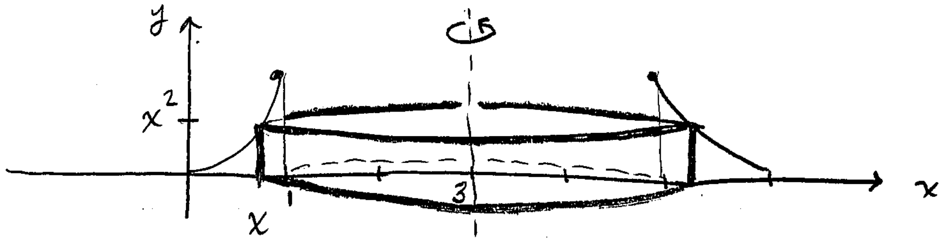
$$\therefore I = \int \frac{2}{x} dx + \int \frac{-x + 3}{x^2 + 1} dx$$

$$= 2 \int \frac{1}{x} dx - \frac{1}{2} \underbrace{\int \frac{2x}{x^2+1} dx}_{\substack{u = x^2 + 1 \\ du = 2x\,dx}} + 3 \int \frac{1}{x^2+1} dx$$

$$= \boxed{2 \ln|x| - \frac{1}{2} \ln|x^2 + 1| + 3 \arctan(x) + C}$$

Question 3:

(a)[5] The region bounded by the curve $y = x^2$, $y = 0$, $x = 0$ and $x = 1$ is rotated about the line $x = 3$. Determine the volume of the resulting solid. The method of cylindrical shells is easiest here.



V_x = volume of cylinder at x

$$= 2\pi \underbrace{(3-x)}_{\text{radius}} \underbrace{x^2}_{\text{height}} \underbrace{dx}_{\text{wall thickness}}$$

$\therefore$ Total volume $V = \int_0^1 2\pi(3-x)x^2\,dx$

$$= 2\pi\left[x^3 - \frac{x^4}{4}\right]_0^1$$

$$= (2\pi)\left(1 - \frac{1}{4}\right)$$

$$= \boxed{\frac{3\pi}{2}}$$

(b)[5] Find the length of the curve $y = x^{3/2}$ from $x = 1$ to $x = 9$.

$$L = \int_1^9 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^9 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$$

$$= \int_1^9 \sqrt{1 + \frac{9}{4}x}\,dx$$

$$= \left[\left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\right]_1^9$$

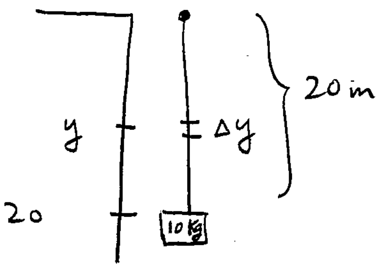
$$= \left(\frac{8}{27}\right)\left[\left(\frac{85}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \left(\frac{1}{27}\right)\left[85^{3/2} - 13^{3/2}\right].$$

Question 4:

(a)[5] 20 m rope hangs over the side of a building and a 10 kg bucket is tied to the end of the rope. A person at the top of the building pulls the rope and bucket up onto the roof of the building. How much work is done if the rope has a total mass of 2 kg? Recall that the acceleration due to gravity is $g = 9.8$ m/s^2.

$$W_{bucket} = (10\ kg)(9.8\ \tfrac{m}{s^2})(20\ m) = 1960\ Nm.$$



$$W_{rope} = \int_{y=0}^{20} \left(\frac{2\ kg}{20 m}\right)(y\ m)\left(9.8\ \frac{m}{s^2}\right)(dy\ m)$$

$$= \frac{9.8}{10}\int_0^{20} y\, dy$$

$$= \frac{9.8}{10}\left[\frac{y^2}{2}\right]_0^{20} = 196\ Nm.$$

$\therefore$ Total work is $W = W_{bucket} + W_{rope}$

$$= 1960 + 196$$

$$= \boxed{2156\ Nm}$$

(b)[5] Solve the following differential equation:

$$\frac{dy}{dx} = \frac{1+y^2}{\sqrt{x+1}}, \quad y(0) = 1$$

You may leave your solution in implicit form (it is not necessary to isolate the y variable in your final answer.)

$$\int \frac{1}{1+y^2}\, dy = \int (x+1)^{-\frac{1}{2}} dx$$

$$\arctan(y) = 2(x+1)^{\frac{1}{2}} + C$$

$\underline{y(0)=1}$:

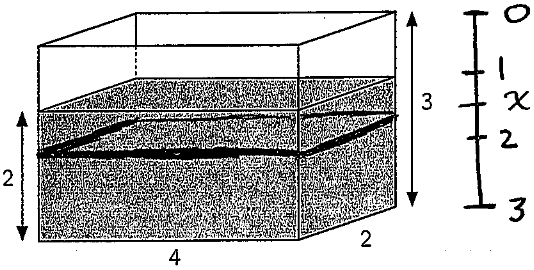
$$\arctan(1) = 2(0+1)^{\frac{1}{2}} + C$$

$$\frac{\pi}{4} = 2 + C$$

$$C = \frac{\pi}{4} - 2$$

$$\therefore \boxed{\arctan(y) = 2(x+1)^{\frac{1}{2}} + \frac{\pi}{4} - 2}$$

Question 5: A rectangular fish tank of length 4 m, width 2 m and height 3 m contains water to a depth of 2 m. Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2.



(a)[5 points] How much work is required to pump all of the water up and out over the top edge of the tank?

Weight of "slice" of water at depth x: $(4)(2)\Delta x \rho g$

Work to lift this slice to top of tank: $(4)(2)\Delta x \rho g\, x$

$= 8\rho g x \Delta x$

$\therefore$ Total work $W = \int_{x=1}^{3} 8\rho g x \, dx$

$$= 8\rho g \left[\frac{x^2}{2}\right]_1^3$$

$$= \frac{8\rho g}{2}[9-1]$$

$$= 32\rho g = (32)(1000)(9.8)$$

$$= \boxed{313{,}600 \text{ Nm}}$$

(b)[5 points] What is the hydrostatic force (force due to water pressure) exerted on one of the long sides of the tank? Recall that pressure P as a function of depth h is $P(h) = \rho g h$ where ρ is the density of the liquid and g is acceleration due to gravity.

area of strip of width Δx is $4\Delta x$.

$\therefore$ force on this strip is $4\Delta x \rho g (x-1)$

$\therefore$ Total force $F = \int_{x=1}^{x=3} 4\rho g (x-1)\, dx$

$$= 4\rho g \left[\frac{x^2}{2} - x\right]_1^3$$

$$= 4\rho g \left[\frac{9}{2} - 3 - \frac{1}{2} + 1\right]$$

$$= 8\rho g$$

$$= (8)(1000)(9.8) = \boxed{78400 \text{ N}}$$

Question 6:

(a)[5 points] Determine the first three non-zero terms of the Maclaurin series for the function $f(x) = x^3 e^{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\therefore\ e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$x^3 e^{x^2} = x^3 + x^5 + \frac{x^7}{2!} + \cdots$$

$\therefore$ first three nonzero terms are $\boxed{x^3 + x^5 + \frac{x^7}{2!}}$

(b)[5 points] Use a Maclaurin series (not L'Hospital's Rule) to evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{x^3(e^x - 1)}$$

$$= \lim_{x \to 0} \frac{1 - \left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \cdots\right]}{x^3\left[1 + x + \frac{x^2}{2!} + \cdots - 1\right]}$$

$$= \lim_{x \to 0} \frac{\left[\frac{x^4}{2!} - \frac{x^8}{4!} + \cdots\right]}{\left[x^4 + \frac{x^5}{2!} + \frac{x^6}{3!} + \cdots\right]} \quad \begin{matrix} \div x^4 \\ \div x^4 \end{matrix}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2!} - \frac{x^4}{4!} + \cdots}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots}$$

$$= \boxed{\frac{1}{2}}$$