Question 1:

(a)[5] Determine $I = \int x \arccos x \, dx$

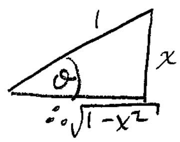
$$\left.\right\} \text{let } u = \arccos(x), \quad dv = x\,dx$$
$$du = \frac{-1}{\sqrt{1-x^2}}, \quad v = \frac{x^2}{2}$$

$$\therefore I = \int u\,dv = uv - \int v\,du = \underbrace{\frac{x^2\arccos(x)}{2}}_{I_0} + \underbrace{\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx}_{I_1}$$

For: $I_1 = \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$, let $x = \sin\theta$, $dx = \cos\theta\,d\theta$.

$$\therefore I_1 = \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta\,d\theta = \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{1}{4}\int 1-\cos(2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin(2\theta)}{2}\right] = \frac{\theta}{4} - \frac{\sin\theta\cos\theta}{4}$$



$$= \frac{1}{4}\arcsin(x) - \frac{1}{4}x\sqrt{1-x^2}$$

$$\therefore I = I_0 + I_1 = \boxed{\frac{x^2\arccos(x)}{2} + \frac{1}{4}\arcsin(x) - \frac{1}{4}x\sqrt{1-x^2} + C}$$

(b)[5] Evaluate $\int_1^\infty \frac{1}{x^3+4x^2}\,dx$

Let $I = \int \frac{1}{x^3+4x^2}\,dx = \int \frac{1}{x^2(x+4)}\,dx.$

$$\frac{1}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4} = \frac{Ax(x+4) + B(x+4) + Cx^2}{x^2(x+4)}$$

$$= \frac{(A+C)x^2 + (4A+B)x + 4B}{x^2(x+4)}.$$

$\therefore\ 4B = 1 \Rightarrow B = \frac{1}{4}.$

$4A + B = 0 \Rightarrow A = -\frac{B}{4} = \frac{-1}{16}.$

$A + C = 0 \Rightarrow C = -A = \frac{1}{16}.$

$$\therefore I = -\frac{1}{16}\int \frac{1}{x}\,dx + \frac{1}{4}\int \frac{1}{x^2} + \frac{1}{16}\int \frac{1}{x+4}\,dx$$

$$= -\frac{1}{16}\ln|x| - \frac{1}{4}\frac{1}{x} + \frac{1}{16}\ln|x+4| + C = \ln\left|\frac{x+4}{x}\right|^{\frac{1}{16}} - \frac{1}{4}\frac{1}{x} + C.$$

$$\therefore \int_1^\infty \frac{1}{x^3+4x^2}\,dx = \lim_{t\to\infty}\left[\ln\left|\frac{x+4}{x}\right|^{\frac{1}{16}} - \frac{1}{4}\frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\Bigg[\underbrace{\left(\ln\left|\frac{t+4}{t}\right|^{\frac{1}{16}} - \frac{1}{4}\frac{1}{t}\right)}_{\longrightarrow 0} - \left(\ln|5|^{\frac{1}{16}} - \frac{1}{4}\right)\Bigg] = \boxed{\frac{1}{4} - \ln(5)^{\frac{1}{16}}}$$

Question 2:

(a)[5] Determine $\int \sec^5 x \tan x \, dx$

$$= \int \sec^4 x \sec x \tan x \, dx \qquad \text{let } u = \sec x, \ du = \sec x \tan x \, dx$$

$$= \int u^4 \, du$$

$$= \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sec^5 x}{5} + C.}$$

(b)[5] Determine $I = \int \frac{x+1}{\sqrt{9-x^2}} \, dx$

$$\therefore I = \underbrace{\int \frac{x}{\sqrt{9-x^2}} \, dx}_{I_1} + \underbrace{\int \frac{1}{\sqrt{9-x^2}} \, dx}_{I_2}.$$

For I_1, let $u = 9 - x^2$, $du = -2x\,dx$.

$$\therefore I_1 = -\frac{1}{2} \int u^{-\frac{1}{2}} \, du = -\frac{1}{2} \frac{u^{1/2}}{(1/2)} = -\sqrt{9-x^2}$$

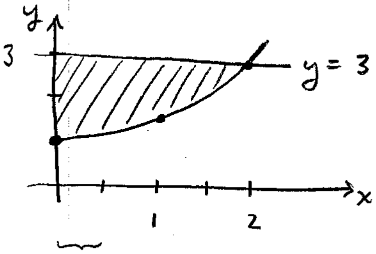
For I_2, let $x = 3\sin\theta$, $dx = 3\cos\theta \, d\theta$

$$\therefore I_2 = \int \frac{1}{\sqrt{9 - 9\sin^2\theta}} 3\cos\theta \, d\theta = \int d\theta = \theta = \arcsin\left(\frac{x}{3}\right).$$

$$\therefore I = I_1 + I_2 = \boxed{-\sqrt{9-x^2} + \arcsin\left(\frac{x}{3}\right) + C.}$$

Question 3:

(a)[4] Let S be the region in the first quadrant that lies between the line $y = 3$ and the curve $y = \sqrt{1+x^3}$. Use Simpson's rule with $n = 4$ to approximate the area of S.



$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$A = \int_0^2 \left[3 - \sqrt{1+x^3}\right] dx.$$

Let $f(x) = 3 - \sqrt{1+x^3}$.

Then $A \approx S_4$

$$= \frac{\Delta x}{3}\left[f(0) + 4f(\tfrac{1}{2}) + 2f(1) + 4f(\tfrac{3}{2}) + f(2)\right]$$

$$= \frac{(1/2)}{3}\left[2 + 4\left(3 - \sqrt{1+(\tfrac{1}{2})^3}\right) + 2\left(3-\sqrt{2}\right) + 4\left(3-\sqrt{1+(\tfrac{3}{2})^3}\right) + 0\right]$$

$$\doteq \boxed{2.76}$$

(b)[3] The function $f(x) = xe^x$ has n^{th} derivative $f^{(n)}(x) = (x+n)e^x$. If using the midpoint rule to approximate $\int_0^2 xe^x\,dx$, how many subintervals are required to ensure that the approximation is accurate to within 0.01?

$$f''(x) = (x+2)e^x.$$

On $[0,2]$, $|f''(x)| = |(x+2)e^x| \leq (2+2)e^2 = 4e^2 = K.$

We require

$$E_{M_n} \leq 0.01$$

$$\therefore \frac{K(b-a)^3}{24n^2} \leq 0.01$$

$$\frac{4e^2(2-0)^3}{24n^2} \leq 0.01$$

$$\frac{4e^2}{3(0.01)} \leq n^2$$

$$\therefore n \geq \sqrt{\frac{4e^2}{3(0.01)}} \approx 31.4$$

$$\therefore \boxed{n = 32}$$

(c)[3] If using the Simpson's rule to approximate $\int_0^2 xe^x\,dx$, how many subintervals are required to ensure that the approximation is accurate to within 0.01?

$$f^{(IV)}(x) = (x+4)e^x,$$

On $[0,2]$, $|f^{(IV)}(x)| = |(x+4)e^x| \leq (2+4)e^2 = 6e^2 = K.$

Require $E_{S_n} \leq 0.01 \Rightarrow \dfrac{K(b-a)^5}{180n^4} \leq 0.01$

$$\frac{6e^2(2-0)^5}{180n^4} \leq 0.01$$

$$\therefore n \geq \left[\frac{6e^2 \cdot 2^5}{180(0.01)}\right]^{1/4} \approx 5.3$$

$$\therefore \boxed{n = 6}$$

Question 4:

(a)[5] Determine with proper justification whether $\int_1^{\infty} \frac{x^2}{x^4 + e^x}\,dx$ converges or diverges.

On $[1, \infty)$ $\quad \frac{x^2}{x^4+e^x} \leq \frac{x^2}{x^4} = \frac{1}{x^2}$.

Since $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges (p- integral, $p>1$),

by the Comparison Theorem, so does $\int_1^{\infty} \frac{x^2}{x^4+e^x}\,dx$.

(b)[5] Evaluate the improper integral $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$. Show all steps including any required limits.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}}\,dx.$$

For $I = \int \frac{\ln x}{\sqrt{x}}\,dx$, let $u = \ln x$, $dv = x^{-\frac{1}{2}}\,dx$

$du = \frac{1}{x}\,dx$, $v = 2x^{\frac{1}{2}}$

$$\therefore I = 2x^{\frac{1}{2}}\ln x - 2\int x^{\frac{1}{2}}\frac{1}{x}\,dx = 2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}}$$
$$= 2x^{\frac{1}{2}}\left[\ln x - 2\right].$$

$$\therefore \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2x^{\frac{1}{2}}\left[\ln x - 2\right]\Big|_t^1$$

$$= \lim_{t\to 0^+}\Big[-4 - \underbrace{2t^{\frac{1}{2}}(\ln t - 2)}_{\text{"}0\cdot(-\infty)\text{"}}\Big]$$: indeterminate form

$$= \lim_{t\to 0^+}\left[-4 - 2\,\frac{(\ln t - 2)}{t^{-\frac{1}{2}}}\right]$$

$$= \lim_{t\to 0^+}\left[-4 - 2\,\frac{\frac{1}{t}}{-\frac{1}{2}t^{-\frac{3}{2}}}\right]$$ by L'Hospital's Rule

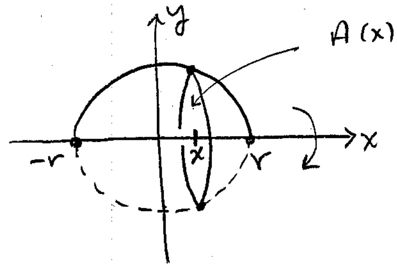
$$= \lim_{t\to 0^+}\left[-4 + 4t^{\frac{1}{2}}\right]$$

$$= \boxed{-4}$$

Question 5:

(a)[5] A sphere of radius r has volume $V = 4\pi r^3/3$. Derive this formula using integration.

Easiest method: rotate $y = \sqrt{r^2 - x^2}$ about x-axis:



$$A(x) = \pi\left(\sqrt{r^2-x^2}\right)^2 = \pi(r^2 - x^2).$$

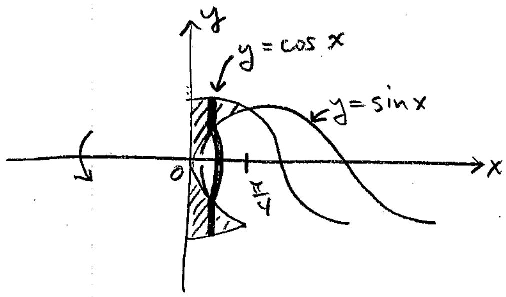
$$\therefore V = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right]$$

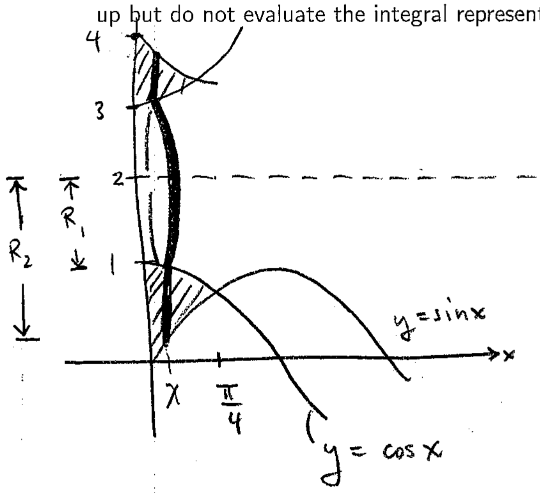
$$= \frac{4\pi r^3}{3}.$$

(b)[2] The region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$ is rotated about the line x-axis. Set up but do not evaluate the integral representing the volume of the resulting solid.



$$\therefore \quad V = \int_0^{\frac{\pi}{4}} \pi\left[\cos^2 x - \sin^2 x\right] dx.$$

(b)[3] The region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$ is rotated about the line $y = 2$. Set up but do not evaluate the integral representing the volume of the resulting solid.



$$V = \int_0^{\frac{\pi}{4}} \pi\left[R_2^2 - R_1^2\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \pi\left[(2-\sin x)^2 - (2-\cos x)^2\right] dx$$