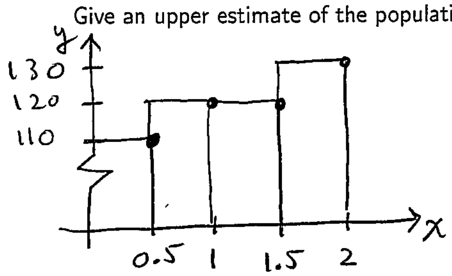
Question 1:

(a)[3] The rate of increase of a growing town's population is determined at five points in time, resulting in the following data:

t (years)	0	0.5	1	1.5	2
$r(t)$ (people per year)	100	110	120	120	130

NOTE: assume $r(t)$ is an increasing function.

Give an upper estimate of the population increase over the period $t = 0$ to $t = 2$.

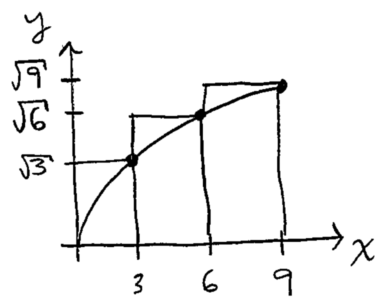


Since $r(t)$ is increasing, population increase $\leq R_4$

where $R_4 = (110)(0.5) + (120)(0.5) + (120)(0.5) + (130)(0.5)$

$= \boxed{240 \text{ people}}$

(b)[3] Use three subintervals and right hand endpoints to estimate the area under the graph of $f(x) = \sqrt{x}$ over the interval $[0, 9]$. Round your answer to one decimal. Is your answer an over-estimate or under-estimate of the true area? Explain briefly.



$\therefore R_3 = (\sqrt{3})(3) + (\sqrt{6})(3) + (\sqrt{9})(3)$

$\approx \boxed{21.5}$

This is an over-estimate since $f(x) = \sqrt{x}$ is increasing, so approximating rectangles extend above the curve.

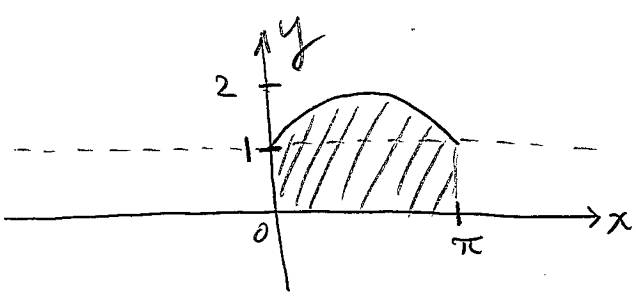
(c)[4] The limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \sin\left(i\frac{\pi}{n}\right)\right] \frac{\pi}{n}$$

represents the area between the graph of a certain function and the x-axis. Draw the graph and shade the area in question. To get full marks you must correctly identify the function and the interval over which the area is measured.

Here $\Delta x = \frac{\pi}{n}$, $a = 0$, $b = \pi$

and $f(x) = 1 + \sin(x)$:



Question 2:

(a)[4] Find the average value of $f(x) = 4x^3 + \frac{2}{x}$ on the interval $[1, e]$.

$$f_{ave} = \frac{1}{e-1} \int_1^e 4x^3 + \frac{2}{x}\, dx$$

$$= \frac{1}{e-1} \left[\frac{\cancel{4}x^4}{\cancel{4}} + 2\ln|x| \right]_1^e$$

$$= \frac{1}{e-1} \left[\left(e^4 + 2\cancel{\ln|e|}\right) - \left(1^4 + \cancel{2\ln|1|}\right) \right]$$

$$= \boxed{\frac{e^4+1}{e-1}}$$

(b)[3] Compute $F'(1)$ if $F(x) = \int_0^{x^2} e^{\sqrt{t}}\, dt$

$$F'(x) = e^{\sqrt{x^2}} \cdot (2x) \quad \text{by FTC I}$$

$$\therefore F'(1) = e^{\sqrt{1^2}}(2)(1) = \boxed{2e}$$

(c)[4] Evaluate $\int_{-\pi/2}^{\pi/2} \sqrt{\sin x + 1} \cos x \, dx$.

Let $u = \sin x + 1$, $du = \cos x\, dx$

$x = -\frac{\pi}{2} \Rightarrow u = \sin\left(-\frac{\pi}{2}\right) + 1 = 0$

$x = \frac{\pi}{2} \Rightarrow u = \sin\left(\frac{\pi}{2}\right) + 1 = 2$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin x + 1} \cos x\, dx = \int_0^2 \sqrt{u}\, du$$

$$= \left[\frac{2}{3} u^{\frac{3}{2}}\right]_0^2$$

$$= \boxed{\frac{2}{3}\, 2^{\frac{3}{2}}}$$

Question 3:

(a)[3] Evaluate $\int \frac{\tan^{-1} x}{1+x^2}\,dx$.

Let $u = \tan^{-1} x$

$du = \frac{1}{1+x^2}\,dx$

$$\therefore \int \frac{\tan^{-1}x}{1+x^2}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \boxed{\frac{(\tan^{-1}x)^2}{2} + C}$$

(b)[4] Suppose $\int_0^5 f'(x)\,dx = 11$, where $f'(x)$ is continuous. If $f(0) = -2$, what is $f(5)$?

$$\int_0^5 f'(x)\,dx = f(5) - f(0)$$

$$\therefore \quad 11 = f(5) - (-2)$$

$$\therefore \boxed{f(5) = 9}$$

(c)[3] The average value of $f(x) = 1 + x$ over the interval $[-2, k]$ is 1. What must be k?

$$1 = \frac{1}{k-(-2)} \int_{-2}^{k} 1+x\,dx \qquad (*)$$

$$k+2 = \left[x + \frac{x^2}{2}\right]_{-2}^{k}$$

$$k+2 = \left[k + \frac{k^2}{2}\right] - \cancel{\left[-2+2\right]} \rightarrow 0$$

$$\cancel{2k}+4 = \cancel{2k} + k^2$$

$\therefore k = 2,\ \cancel{-2} \longleftarrow$ because of denominator in $(*)$

$$\therefore \boxed{k=2}$$

Question 4:

(a)[4] Evaluate $\int \frac{e}{t\ln(t)}\,dt$.

Let $u = \ln(t)$

$du = \frac{1}{t}\,dt$

$$\therefore \int \frac{e}{t\ln(t)}\,dt = e\int \frac{1}{u}\,du$$

$$= e\ln|u| + C$$

$$= \boxed{e\ln\big|\ln(t)\big| + C}$$

(b)[3] Evaluate $\int \frac{x}{1+x}\,dx$.

Let $u = 1+x$

$du = dx$

so $x = u-1$

$$\therefore \int \frac{x}{1+x}\,dx = \int \frac{u-1}{u}\,du$$

$$= \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| + C$$

$$= \boxed{1+x-\ln|1+x| + C}$$

(c)[4] Evaluate $\int \sin^2(2x)\,dx$. (Hint: recall that $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$.)

$$I = \int \sin^2(2x)\,dx = \int \frac{1-\cos(4x)}{2}\,dx = \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos(4x)\,dx$$

$$= I_1 - I_2 \text{ ; say .}$$

$$I_1 = \int \frac{1}{2}\,dx = \frac{1}{2}x$$

For I_2, let $u = 4x$, $du = 4\,dx$.

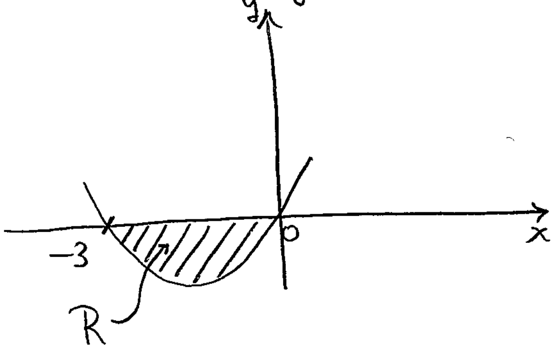
$$\therefore I_2 = \frac{1}{2}\cdot\frac{1}{4}\int \cos(u)\,du = \frac{1}{8}\sin(u) + C = \frac{1}{8}\sin(4x) + C$$

$$\therefore I = I_1 - I_2 = \boxed{\frac{1}{2}x - \frac{1}{8}\sin(4x) + C}$$

Question 5:

(a)[4] Without evaluating the integral, explain why $\int_{-3}^{0}(x^2+3x)\,dx = 3$ can not be correct.

The integral corresponds to the region R:



Since R is located entirely below the x-axis,

$$\int_{-3}^{0}(x^2+3x)dx < 0,$$

so $\int_{-3}^{0}(x^2+3x)dx \neq 3$.

(b)[3] Let $f(x) = \int_{x}^{3x} \frac{1}{t}\,dt$. Show that $f(x)$ is constant on $(0,\infty)$.

Sol^n 1:

$$\int_{x}^{3x}\frac{1}{t}dt$$

$$= \Big[\ln|t|\Big]_{x}^{3x}$$

$$= \ln|3x| - \ln|x|$$

$$= \ln(3x) - \ln(x) \quad \text{since } x>0$$

$$= \ln 3 + \cancel{\ln x} - \cancel{\ln x}$$

$$= \ln 3, \text{ a constant}$$

Sol^n 2:

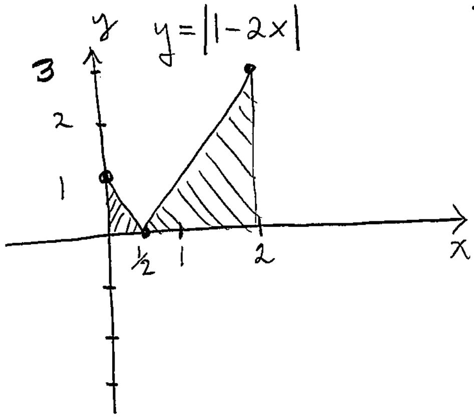
$$f'(x) = \frac{1}{\cancel{3}x}\cdot\cancel{3} - \frac{1}{x}\cdot 1 \quad \text{by FTC I}$$

$$= \frac{1}{x} - \frac{1}{x}$$

$$= 0$$

Since $f'(x)=0$ on $(0,\infty)$, f must be constant.

(c)[4] Use an area interpretation to evaluate $\int_{0}^{2}|1-2x|\,dx$.



$$\therefore \int_{0}^{2}|1-2x|dx = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)(1) + \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)(3)$$

$$= \frac{1}{4} + \frac{9}{4}$$

$$= \boxed{\frac{5}{2}}$$