# 1 Sequences

## 1.1 Overview

A (numerical) **sequence** is a list of real numbers in which each entry is a function of its position in the list. The entries in the list are called **terms**. For example,

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$$

is a sequence with first term $1$, second term $1/2$, third term $1/3$, etc. A sequence is typically denoted $\{a_n\}_{n=1}^{\infty}$, where the **subscript** $n$, called the **index**, indicates the position of the term $a_n$ in the list. That is,

$$\{a_n\}_{n=1}^{\infty} = \underset{\substack{\uparrow \\ \text{1}^{\text{st}} \\ \text{term}}}{a_1,} \quad \underset{\substack{\uparrow \\ \text{2}^{\text{nd}} \\ \text{term}}}{a_2,} \quad \underset{\substack{\uparrow \\ \text{3}^{\text{rd}} \\ \text{term}}}{a_3,} \quad \ldots$$

The terms of a sequence are often given as a formula, which gives us the "recipe" for the sequence. For example, the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ written out is

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$$

Here, the general $n^{\text{th}}$ term is $a_n = 1/n$, so $a_1 = 1/1$, $a_2 = 1/2$, and so on.

The index does not always start at $n = 1$. For that matter, the index need not be denoted with the letter $n$. For example,

$$\begin{aligned}\{2^k\}_{k=0}^{\infty} &= 2^0,\ 2^1,\ 2^2,\ 2^3,\ \ldots \\ &= 1,\ 2,\ 4,\ 8,\ \ldots\end{aligned}$$

Here, $a_k = 2^k$, $k \geq 0$ .

Here's another example:

**Example:** Define a sequence by $b_k = k/(1 + 2^k)$, $k = 1, 2, 3, \ldots$. Write down the first three terms of the sequence.

**Solution:**

$$\begin{aligned}&b_1,\ b_2,\ b_3 \\ =&\frac{1}{1+2^1},\ \frac{2}{1+2^2},\ \frac{3}{1+2^3} \\ =&\frac{1}{3},\ \frac{2}{5},\ \frac{1}{3}\end{aligned}$$

□

We are interested in two specific types of sequences: **(i) arithmetic** and **(ii) geometric**.

## 1.2 Arithmetic Sequences

**Definition:** A sequence $a_1,\ a_2,\ a_3,\ \ldots$ with the property that

$$\begin{aligned} a_2 - a_1 &= d \\ a_3 - a_2 &= d \\ a_4 - a_3 &= d \\ &\vdots \\ a_n - a_{n-1} &= d \\ &\vdots \end{aligned}$$

is called an **arithmetic sequence** with **common difference** $d$.

In simple terms, an arithmetic sequence is characterized by the property that the difference between consecutive terms is the same. An arithmetic sequence is also called an **arithmetic progression**.

**Example:** Let $a_n = 5 - 2n,\ n = 1, 2, 3, \ldots$

*(i)* List the first three terms of the sequence.

*(ii)* Is the sequence arithmetic?

*(iii)* If yes to *(ii)*, find the common difference.

**Solution:**

*(i)* The first three terms are

$$\begin{aligned} & a_1,\ a_2,\ a_3 \\ =& 5 - 2(1),\ 5 - 2(2),\ 5 - 2(3) \\ =& 3,\ 1,\ -1 \end{aligned}$$

*(ii)* Suppose $k \geq 1$ is any positive integer. Then

$$\begin{aligned} & a_{k+1} - a_k \\ =& [5 - 2(k+1)] - [5 - 2(k)] \\ =& 5 - 2k - 2 - 5 + 2k \\ =& -2 \end{aligned}$$

Since $k$ was arbitrary, we conclude that the difference between any two consecutive terms is $-2$, and so the sequence is arithmetic.

*(iii)* From *(ii)* we conclude that the common difference is $d = -2$. □

**Example:** Suppose $\{a_n\}_{n=1}^{\infty}$ is an arithmetic sequence with first term $3$ and common difference $7/3$. Find a formula for $a_n$.

**Solution:** Since the common difference is $7/3$ and the first term is $3$, write out the first few terms to establish a pattern:

$$\begin{aligned} & a_1,\ a_2,\ a_3,\ a_4,\ \ldots \\ =& 3,\ 3+7/3,\ 3+7/3+7/3,\ 3+7/3+7/3+7/3,\ \ldots \\ =& 3,\ 3+7/3,\ 3+2(7/3),\ 3+3(7/3),\ \ldots \end{aligned}$$

By inspection, we see that term $n$ has form $3+(n-1)(7/3)$. That is, $a_n = 3+(n-1)(7/3)$. $\square$

This last example generalizes to give a standard form for arithmetic sequences: an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ with first term $a$ and common difference $d$ has $n^{\text{th}}$ term $a_n = a+(n-1)d$.

**Example:** An arithmetic sequence has $4^{\text{th}}$ term $14/15$ and $11^{\text{th}}$ term $7/3$. Find a formula for $a_n$.

**Solution:** Since $a_4 = a+(4-1)d = a+3d$ and $a_{11} = a+(11-1)d = a+10d$ we have

$$\begin{aligned} a+10d &= \frac{7}{3} \\ a+3d &= \frac{14}{15}\ . \end{aligned}$$

Subtracting left hand sides and right hand sides respectively gives

$$\begin{aligned} (a+10d)-(a+3d) &= \frac{7}{3}-\frac{14}{15} \\ 7d &= \frac{7}{5} \\ \text{so}\ \ d &= \frac{1}{5}\ . \end{aligned}$$

Now substitute this value for $d$ into one of the expressions for the given terms of the series, for example

$$\begin{aligned} a+10d &= \frac{7}{3} \\ a+10\left(\frac{1}{5}\right) &= \frac{7}{3} \\ a &= \frac{7}{3}-2 \\ a &= \frac{1}{3}\ . \end{aligned}$$

Therefore, $a_n = \dfrac{1}{3}+(n-1)\dfrac{1}{5}$ . $\square$

## 1.3 Geometric Sequences

The general development of geometric sequences parallels that of arithmetic sequences, except that we consider division by a common value rather than addition:

**Definition:** A sequence $a_1,\ a_2,\ a_3,\ \ldots$ with the property that

$$\begin{aligned}\frac{a_2}{a_1} &= r\\ \frac{a_3}{a_2} &= r\\ \frac{a_4}{a_3} &= r\\ &\vdots\\ \frac{a_n}{a_{n-1}} &= r\\ &\vdots\end{aligned}$$

is called a **geometric sequence** with **common ratio** $r$.

For a geometric sequence, the ratio of consecutive terms is the same. A geometric sequence is also called a **geometric progression**.

**Example:** Let $\{b_n\}_{n=1}^{\infty} = \left\{\frac{3}{7^{n-1}}\right\}_{n=1}^{\infty}$.

*(i)* List the first three terms of the sequence.

*(ii)* Is the sequence geometric?

*(iii)* If yes to *(ii)*, find the common ratio.

**Solution:**

*(i)* The first three terms are

$$\begin{aligned}&b_1,\ b_2,\ b_3\\ =&\frac{3}{7^{1-1}},\ \frac{3}{7^{2-1}},\ \frac{3}{7^{3-1}}\\ =&3,\ \frac{3}{7},\ \frac{3}{49}\end{aligned}$$

*(ii)* Suppose $k \geq 1$ is any positive integer. Then

$$\begin{aligned}&\frac{b_{k+1}}{b_k}\\ =&\frac{3}{7^{k+1-1}} \Big/ \frac{3}{7^{k-1}}\\ =&\frac{3}{7^k}\frac{7^{k-1}}{3}\\ =&\frac{1}{7}\end{aligned}$$

Since $k$ was arbitrary, we conclude that $b_{k+1}/b_k = 1/7$ for all integers $k \geq 1$, so that the sequence is geometric.

*(iii)* From *(ii)* we conclude that the common ratio is $r = 1/7$. □

**Example:** Suppose $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence with first term $a$ and common ratio $r$. Find a formula for $a_n$.

**Solution:** Since the common ratio is $r$ and the first term is $a$, the sequence has the form

$$\begin{aligned}&a_1,\ a_2,\ a_3,\ a_4,\ \ldots\\ =&a,\ a\cdot r,\ a\cdot r\cdot r,\ a\cdot r\cdot r\cdot r,\ \ldots\\ =&a,\ ar,\ ar^2,\ ar^3,\ \ldots\end{aligned}$$

By inspection, we see that term $n$ has form $ar^{n-1}$. That is, $a_n = ar^{n-1}$. □

This last example leads us to conclude: a geometric sequence $\{a_n\}_{n=1}^{\infty}$ with first term $a$ and common ratio $r$ has $n^{\text{th}}$ term $a_n = ar^{n-1}$.

**Example:** A geometric sequence has $a_3 = 7/4$ and $a_6 = -7/32$. Find $a_8$.

**Solution:** We have $a_3 = ar^{3-1} = ar^2$ and $a_6 = ar^{6-1} = ar^5$, so

$$\begin{aligned}ar^5 &= \frac{-7}{32}\\ ar^2 &= \frac{7}{4}\ .\end{aligned}$$

Dividing left and right hand sides respectively gives

$$\begin{aligned}\frac{ar^5}{ar^2} &= \frac{\left(\frac{-7}{32}\right)}{\left(\frac{7}{4}\right)}\\ r^3 &= \frac{-1}{8}\\ \text{so}\ \ r &= \frac{-1}{2}\ .\end{aligned}$$

Now substitute this value for $r$ into one of the expressions for the given terms of the series, for example

$$\begin{aligned} ar^2 &= \frac{7}{4} \\ a\left(\frac{-1}{2}\right)^2 &= \frac{7}{4} \\ a\left(\frac{1}{4}\right) &= \frac{7}{4} \\ \text{so} \quad a &= 7 \ . \end{aligned}$$

Finally,

$$\begin{aligned} a_8 &= ar^7 \\ &= 7\left(\frac{-1}{2}\right)^7 \\ &= \frac{-7}{128} \end{aligned}$$

□

## 1.4 Problems

1. Write the first four terms of the sequence defined by $a_n = \dfrac{2n-1}{n^2+2n}$, $n \geq 1$.

   ans: $1/3,\ 3/8,\ 1/3,\ 7/24$

2. For the sequence $\{a_n\}_{n=1}^{\infty}$ with first five terms $\sqrt{2},\ 2,\ \sqrt{6},\ 2\sqrt{2},\ \sqrt{10}$, give a possible expression for $a_n$.

   ans: $\sqrt{2n}$

3. Find an expression for $a_n$ for the arithmetic sequence $3/5,\ 1/10,\ -2/5,\ \ldots$

   ans: $3/5 - (n-1)5/10$

4. Find the $14^{\text{th}}$ term of the arithmetic sequence $3,\ 7/3,\ 5/3,\ \ldots$

   ans: $-17/3$

5. An arithmetic sequence has $a_{17} = 25/3$ and $a_{32} = 95/6$. What is $a_6$?

   ans: $17/6$

6. If $a_1 = 25$, $d = -14$ and $a_n = -507$ then what is $n$ if the sequence is arithmetic?

   ans: $39$

7. Find the $23^{\text{rd}}$ term of the geometric sequence $7/625,\ -7/25,\ \ldots$

   ans: $7 \cdot 5^{40}$

8. Find an expression for $a_n$ for the geometric sequence $2/x,\ 4/x^2,\ \ldots$

ans: $(2/x)^n$

9. If a geometric sequence has $a_4 = 8/3$ and $a_7 = 64/3$, what is $a_5$?

ans: $16/3$

10. A geometric sequence has the property that $a_{n+3} = 27a_n$. What then is $r$?

ans: $3$

# 2 Series

Now that we have studied sequences in general, and arithmetic and geometric sequences in particular, we make use of these to define the notion of a **series**. Given a sequence

$$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots$$

we define the associated series to be sum of the terms of the sequence. A series may be **finite**, in which case we use $S_n$ to denote the sum of the first $n$ terms of the sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n \ .$$

For example, given the sequence defined by $a_n = 1/2^n$, $n = 1, 2, 3, \ldots$, we may form the series of the first $100$ terms

$$\begin{aligned} S_{100} &= a_1 + a_2 + a_3 + \cdots + a_{100} \\ &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{100}} \end{aligned}$$

We will be concerned with finite series, and in particular how to add them up. We will find convenient expressions for $S_n$ which allow us to easily determine the sum without having to add the terms one at a time.

Note that a series may also be **infinite**, meaning we continue to add terms without stopping. For an infinite series there is no final term in the sum. An example of an infinite series is

$$S = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

To most people, the idea of adding up an infinite number of terms is difficult to grasp, as surely the resulting sum must be infinite. This is not always the case however, and there is a well defined mathematical procedure for determining the sum of an infinite series when it exists. For example, it turns out that

$$\begin{aligned} S &= \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \\ &= 1 \end{aligned}$$

Another well known infinite series is

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

which, quite astoundingly, adds up to $S = \pi^2/6$. We will will not be looking at infinite series further in this course.

We will be concerned with two specific types of series related to our earlier work: **arithmetic series** and **geometric series**.

## 2.1 Arithmetic Series

Given an arithmetic sequence with first term $a$ and common difference $d$, recall the $n^{\text{th}}$ term had the form

$$a_n = a + (n-1)d, \qquad n = 1, 2, 3, \ldots$$

We define the $n$-term arithmetic series as the sum of the first $n$ terms of the arithmetic sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

There is a very simple formula for $S_n$ which we will now derive. Write

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= [a + 0d] + [a + 1d] + [a + 2d] + \cdots + [a + (n-2)d] + [a + (n-1)d] \ , \end{aligned}$$

where in the last line above we have written out the terms of the series in their standard form $a_n = a + (n-1)d$. Now reverse the order of the terms in the last line above and write them in the following convenient arrangement:

$$\begin{aligned} S_n &= [a + 0d] \quad &+[a + 1d] + [a + 2d] \quad &+ \cdots + [a + (n-2)d] \quad &+[a + (n-1)d] \\ S_n &= [a + (n-1)d] \quad &+[a + (n-2)d] + [a + (n-3)d] \quad &+ \cdots + [a + 1d] \quad &+[a + 0d] \ . \end{aligned}$$

Now add both equations by adding down the columns as we've written them:

$$2S_n = [a + a + (n-1)d] \quad +[a + a + (n-1)d] + \cdots \quad +[a + a + (n-1)d] + [a + a + (n-1)d] \ .$$

The right hand side of this last equation is the sum of the $n$ terms $[a + a + (n-1)d]$, and so

$$2S_n = n[2a + (n-1)d]$$

which gives

$$\boxed{S_n = \frac{n[2a + (n-1)d]}{2} \ .}$$

Using this equation, another convenient form for $S_n$ is found by writing the term in square brackets as terms of the sum:

$$\begin{aligned} S_n &= \frac{n[2a + (n-1)d]}{2} \\ &= \frac{n[a + a + (n-1)d]}{2} \end{aligned}$$

So that

$$\boxed{S_n = n\frac{(a_1 + a_n)}{2}\ .}$$

This last expression for $S_n$ says that the sum of the first $n$ terms of an arithmetic series is the average of the first and last terms, multiplied by the number of terms.

**Example:** What it sum of the first $n$ positive integers?

**Solution:** The question is to find

$$S_n = 1 + 2 + 3 + \cdots + n\ .$$

The sequence $1, 2, 3, \ldots, n$ is arithmetic with $a = 1$ and $d = 1$, and we have $a_1 = 1$ and $a_n = n$. Thus

$$\begin{aligned} S_n &= n\frac{(a_1 + a_n)}{2} \\ &= n\frac{1 + n}{2} \\ &= \frac{n(n+1)}{2} \end{aligned}$$

□

**Example:** What it sum of the first $n$ odd positive integers?

**Solution:** This time the sum is

$$S_n = 1 + 3 + 5 + \cdots + a_n\ .$$

The sequence of odd integers is arithmetic with $a = 1$ and $d = 2$, and so $a_n = 1 + (n-1)2 = 2n - 1$. Therefore

$$\begin{aligned} S_n &= n\frac{(a_1 + a_n)}{2} \\ &= n\frac{(1 + 2n - 1)}{2} \\ &= n\frac{2n}{2} \\ &= n^2 \end{aligned}$$

□

**Example:** A well drilling company charges $\$16$ to drill the first metre of a well, $\$16.30$ for the next metre, $\$16.60$ for the next, and so, the cost increasing by $\$0.30$ per metre. How much will it cost to drill a $30$ metre well?

**Solution:** Let

$$\begin{aligned} c_n &= \text{cost of drilling the } n^{\text{th}} \text{ metre} \\ &= 16 + (n-1)(0.30)\ . \end{aligned}$$

Notice $c_1, c_2, c_3, \ldots$ is an arithmetic sequence, and the cost to drill 30 metres is $S_{30} = c_1 + c_2 + \cdots + c_{30}$, an arithmetic series with 30 terms. Therefore,

$$\begin{aligned} S_{30} &= 30\frac{(c_1 + c_{30})}{2} \\ &= 30\frac{[16 + (16 + (30-1)(0.30))]}{2} \\ &= \$610.50 \end{aligned}$$

and so it will cost $\$610.50$ to drill the 30 metre well. □

## 2.2 Geometric Series

Given a geometric sequence with first term $a$ and common ratio $r$, recall the $n^{\text{th}}$ term had the form

$$a_n = ar^{n-1}, \qquad n = 1, 2, 3, \ldots$$

We define the $n$-term geometric series as the sum of the first $n$ terms of the geometric sequence:

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n \ .$$

As with the arithmetic series, there is a very simple formula for $S_n$. First write

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \end{aligned}$$

Now multiply both sides of this equation by $r$, giving

$$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n \ ,$$

and subtract the second equation from the first:

$$\begin{aligned} S_n - rS_n = &(a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}) \\ &- (ar + ar^2 + \cdots + ar^{n-1} + ar^n) \ . \end{aligned}$$

Notice that if the bracketed terms on the right hand side are simplified, all terms cancel except $a$ and $ar^n$. Also, the $S_n$ is common to both terms on the left. Thus, this last equation simplifies to

$$S_n(1 - r) = a - ar^n \ ,$$

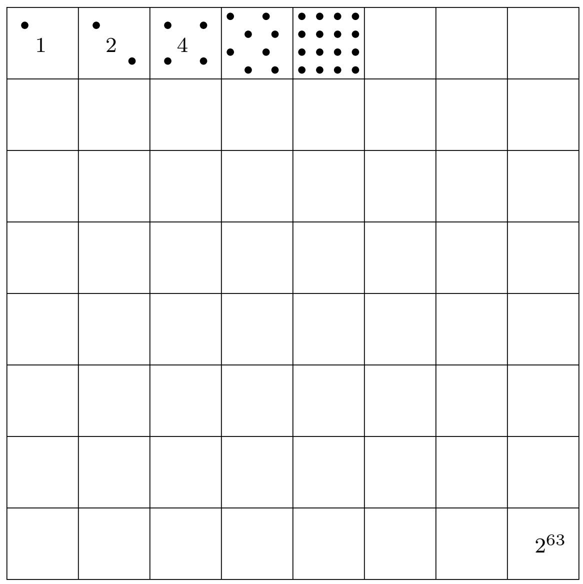
and so the sum of the first $n$ terms of the geometric series is

$$\boxed{S_n = \frac{a(1 - r^n)}{1 - r} \ .}$$

Sums of geometric series arise in many interesting examples; here are a few:

**Example:** The game of chess is played on an $8 \times 8$ board of $64$ squares. There is a legend that the game was invented for a king, and the king was so pleased with the new game that he wanted to reward the inventor, so he asked the inventor what he would like. The inventor replied that he would like one grain of rice for the first square of the chess board, two for the second square, four for the third, and so on, doubling the number of grains for each square of the board for all $64$ squares. The king saw that this was a fairly modest request and so ordered that the rice be delivered to the inventor. If a single grain of rice has a mass of $0.00002$ kilograms, how much rice (in kilograms) was required to deliver on the king's order?

**Solution:** The chessboard with the grains of rice for the first few squares is shown below:




Let $N$ be the total number of grains required:

$$N = 1 + 2 + 4 + \cdots + 2^{63} \ .$$

Notice that the sequence $1, 2, 4, \ldots$ is geometric with $a = 1$ and $r = 2$, and we wish to sum the first $n = 64$ terms of this sequence. Thus

$$\begin{aligned} N &= \frac{a(1 - r^n)}{1 - r} \\ &= \frac{1(1 - 2^{64})}{1 - 2} \\ &= 2^{64} - 1 \ . \end{aligned}$$

The total mass of rice is the number of grains multiplied by the mass of a single grain:

$$\begin{aligned} N \times 0.00002 &= (2^{64} - 1)(0.00002) \\ &\doteq 3.7 \times 10^{14} \text{ kilograms.} \end{aligned}$$

Needless to say, this is a rather large amount of rice. Expressed in scientific notation, the enormity of this quantity may be difficult to appreciate. Here is the number written out:

$$\begin{aligned} N \times 0.00002 &= 368{,}934{,}881{,}474{,}191.032\,3 \text{ kilograms} \\ &= 368{,}934{,}881{,}474.191\,032\,3 \text{ tonnes} \end{aligned}$$

That is, about $369$ billion metric tonnes of rice would be required, or about $369$ trillion $1$ kg bags of rice.
□

## Compound Interest

A very important application of geometric series is that of compound interest. In general terms, when money is lent to a borrower by a lender, interest is paid by the borrower in exchange for the use of the lender's money. For example, suppose you find $\$5000$ in your sock drawer, and you decide to take it to your bank and purchase a type of investment called a GIC (Guaranteed Investment Certificate). Under the terms of the GIC, you give your $\$5000$ to the bank, and at the end of ten years they agree to return the $\$5000$ to you along with earned interest. In this case, you are the lender and the bank is the borrower. In many examples, the "lender" is an investor depositing money into an investment, and the investment can be considered as the "borrower", the expectation being that the investment will return the original amount invested plus interest.

Interest is normally quoted as a percentage. For example, $\$1000$ invested for one year at an annual rate of interest of $7\%$ accumulates to $\$1070$ after one year. The $\$1070$ consists of the original $\$1000$ plus the interest of $(7\%)(\$1000) = (0.07)(\$1000) = \$70$.

Compound interest is the process of adding earned interest to the original amount of the investment, so that this new total (compound) amount then in turn earns interest. That is, interest is then earned on previously earned interest. The compounding frequency states how many times per year the earned interest is added to the interest bearing portion of the investment. In our case we will stick to the simplest type of compound interest: interest compounded annually.

Let's return to our GIC example to see how this works. Suppose you decide to invest $\$5000$ in a $10$ year GIC which pays interest at the rate of $4\%$ compounded annually. Here is a table illustrating the accumulation of interest in the investment over the years:

| Year | value at start of year ($) | $\left\{\text{value at start of year}\right\} + \left\{\text{interest earned in year}\right\} = \left\{\text{value at end of year (\$)}\right\}$ |
|---|---|---|
| 1 | $5000$ | $5000 + (0.04)(5000) = 5000(1.04)$ |
| 2 | $5000(1.04)$ | $5000(1.04) + (0.04)5000(1.04) = 5000(1.04)^2$ |
| 3 | $5000(1.04)^2$ | $5000(1.04)^2 + (0.04)5000(1.04)^2 = 5000(1.04)^3$ |
| ⋮ | ⋮ | ⋮ |
| 10 | $5000(1.04)^9$ | $5000(1.04)^9 + (0.04)5000(1.04)^9 = 5000(1.04)^{10}$ |

Notice how, in each year, the value of the investment at the end of the year is the value at the beginning plus the interest earned in the year. The accumulated value of the investment at the end of one year then becomes the amount subject to interest at the beginning of the next.

Also notice the form of the accumulated value of the investment at the end of each year. This leads us to conclude: An amount $\$x$ invested for $n$ years at a rate of interest of $i\%$ compounded annually has an accumulated value at the end of $n$ years of

$$\boxed{x\left(1+\frac{i}{100}\right)^n .}$$

**Example:** If, on every birthday beginning with the $21^{\text{st}}$ and ending with the $65^{\text{th}}$, a person deposits $\$1000$ into a retirement fund paying $10\%$ interest compounded annually, what will the accumulated value of the fund be on the person's $65^{\text{th}}$ birthday?

**Solution:**

\$1000 deposited on $21^{\text{st}}$ birthday accumulates to $1000(1+0.1)^{44}$
\$1000 deposited on $22^{\text{nd}}$ birthday accumulates to $1000(1+0.1)^{43}$
\$1000 deposited on $23^{\text{rd}}$ birthday accumulates to $1000(1+0.1)^{42}$
$\vdots$
\$1000 deposited on $64^{\text{th}}$ birthday accumulates to $1000(1+0.1)^{1}$
\$1000 deposited on $65^{\text{th}}$ birthday accumulates to $1000(1+0.1)^{0}=1000$

The value of the fund at age $65$ will be total of all of these accumulated amounts:

$$1000+1000(1.1)^2+1000(1.1)^3+\cdots+1000(1.1)^{44} ,$$

a geometric series. Here $a=1000$, $r=1.1$ and $n=45$, and so the total value is given by

$$\begin{aligned}\frac{a(1-r^n)}{1-r} &= 1000\frac{1-(1.1)^{45}}{1-1.1}\\ &\doteq \$718{,}905\end{aligned}$$

Notice here that only $\$45{,}000$ was paid into the retirement fund over the years, yet the fund is worth $\$718{,}905$ at age $65$. □

## 2.3 Problems

1. Find the sum of the first $20$ terms of the series $5+8+11+14+\cdots+a_{20}$

ans: 670

2. Find the sum of the first $300$ even positive integers.

ans: 90,300

3. Find the sum of the first $50$ terms of the arithmetic sequence that has first term $-8$ and $50^{\text{th}}$ term 139.

ans: $3275$

4. Compute $\frac{\pi}{3} + \frac{2\pi}{3} + \pi + \frac{4\pi}{3} + \cdots + \frac{13\pi}{3}$.

ans: $91\pi/3$

5. Compute $\frac{1}{e} + \frac{3}{e} + \frac{5}{e} + \cdots + \frac{21}{e}$.

ans: $121/e$

6. A theatre is designed with $28$ seats in the front row, $32$ in the second, $36$ in the third, and so on, all the way back to the $20^{\text{th}}$ and final row. How many seats are there in the theatre?

ans: $1320$

7. Find the sum of the first nine terms of the geometric series $\frac{1}{18} - \frac{1}{6} + \frac{1}{2} + \cdots$

ans: $4921/18$

8. Find the sum of the geometric series $1 + \sqrt{2} + 2 + \cdots + 32$ .

ans: $63 + 31\sqrt{2}$

9. You convince your employer to pay you $1$¢ for the first day of a month worked, $2$¢ for the next, $4$¢ for the next, and so on, doubling with each day worked. If you work each day of the month of March ($31$ days), how much will you earn?

ans: $\$21,474,836.47$

10. To save for university, a parent makes a $\$1200$ deposit on January 1 of each year to a savings fund which earns interest at $6\%$ compounded annually. If $18$ deposits are made in total, what is the accumulated value of the fund at the time when the last deposit is made?

ans: $\$37,087$

11. Parents of a newborn wish to accumulate $\$150,000$ in an education fund by making equal deposits starting on the day of the child's birth and ending with the $18^{\text{th}}$ birthday. If the education fund pays $8\%$ interest compounded annually, how large should the payments be in order to reach the goal?

ans: $\$3619.14$