Question 1 [10 points]:

(a) [5] The one to one function $f(x) = \sqrt{\frac{2x-1}{5}} + 3$ has domain $[1/2, \infty)$ and range $[3, \infty)$. Determine $f^{-1}(x)$ and state its domain and range.

$$y = \sqrt{\frac{2x-1}{5}} + 3$$

$$x = \sqrt{\frac{2y-1}{5}} + 3$$

$$x - 3 = \sqrt{\frac{2y-1}{5}}$$

$$(x-3)^2 = \frac{2y-1}{5}$$

$$5(x-3)^2 = 2y-1$$

$$y = \frac{5(x-3)^2+1}{2}$$

$$\boxed{\therefore\ f^{-1}(x) = \frac{5(x-3)^2+1}{2}, \quad \text{domain: } [3, \infty) \quad \text{range: } [\tfrac{1}{2}, \infty)}$$

(b) [3] Express as a single logarithm: $\log_2 x + 3\log_2(x^2+1) - \log_2 5$

$$= \log_2 x + \log_2 (x^2+1)^3 - \log_2 5$$

$$= \log_2 x(x^2+1)^3 - \log_2 5$$

$$= \boxed{\log_2 \left[\frac{x(x^2+1)^3}{5}\right]}$$

(c) [2] Calculate $\log_{\sqrt{2}} 7$ (round your answer to two decimal places).

$$\log_{\sqrt{2}} 7 = \frac{\ln 7}{\ln \sqrt{2}} \approx \boxed{5.61}$$

Question 2 [10 points]: Solve the following equations for x:

(a) [5] $2^{3x^2} = 4 \cdot 2^{5x}$

$$2^{3x^2} = 2^2\, 2^{5x}$$

$$2^{3x^2} = 2^{5x+2}$$

$$\therefore\ 3x^2 = 5x+2$$

$$3x^2-5x-2=0$$

$$3x^2-6x+x-2=0$$

$$3x(x-2)+(x-2)=0$$

$$(3x+1)(x-2)=0$$

$$3x+1=0\ ,\quad x-2=0$$

$$\boxed{x=\frac{-1}{3}\ ,\quad x=2}$$

(b) [5] $\ln(x) = \ln(35) - \ln(x-2)$

$$\ln(x)+\ln(x-2)=\ln(35)$$

$$\ln x(x-2)=\ln(35)$$

$$x(x-2)=35$$

$$x^2-2x-35=0$$

$$(x-7)(x+5)=0$$

$$\therefore\ x=7\ ,\quad x=-5$$

<u>Check</u>: $x=7$:

$$\ln(7) \mid \ln(35)-\ln(7-2)$$

$$\ln(7) \mid \ln(35)-\ln(5)$$

$$\ln(7) \mid \ln\left(\frac{35}{5}\right)$$

$$\ln(7) = \ln(7)\ \checkmark$$

<u>$x=-5$</u>: $\underbrace{\ln(-5)}_{\text{not defined!}} \mid \ln(35)-\ln(-5-2)$

$$\boxed{\therefore\ x=7 \text{ is the only solution}}$$

Question 3 [10 points]:

(a) [5] The half-life of iodine-131 is eight days. A sample originally containing 10 g of iodine-131 now contains only 0.3 g. How old is it? Round your answer to two decimal places. (Recall that the amount A of radioactive material present at time t is given by $A(t) = A_0e^{kt}$ where A_0 is the original amount of radioactive material and k is a negative number.)

$$A(t) = A_0 e^{kt}$$

$$A(8) = \tfrac{1}{2}A_0, \text{ so } \tfrac{1}{2}\cancel{A_0} = \cancel{A_0}e^{k\cdot 8}$$

$$\therefore k = \frac{1}{8}\ln\left(\frac{1}{2}\right)$$

Now solve $$0.3 = 10\, e^{\left[\frac{1}{8}\ln\left(\frac{1}{2}\right)\right]t}$$

$$\therefore \frac{0.3}{10} = e^{\frac{1}{8}\ln\left(\frac{1}{2}\right)t}$$

$$\ln\left(\frac{0.3}{10}\right) = \frac{1}{8}\ln\left(\frac{1}{2}\right)t$$

$$\therefore t = \frac{8\ln\left(\frac{0.3}{10}\right)}{\ln\left(\frac{1}{2}\right)}$$

$$\boxed{t \approx 40.47 \text{ days}}$$

(b) [5] The population of one country grows according to the model $P_1(t) = 12{,}000e^{0.02t}$, while that of a second country grows according to the model $P_2(t) = 4000e^{0.025t}$. Here t represents time in years. If $t = 0$ corresponds to the present, in how many years time will the two populations be equal?

Solve $$12000\, e^{0.02t} = 4000\, e^{0.025t}$$

$$3 = \frac{e^{0.025t}}{e^{0.02t}}$$

$$3 = e^{0.005t}$$

$$\ln 3 = 0.005t$$

$$t = \frac{\ln 3}{0.005}$$

$$\boxed{t \approx 219.72 \text{ years.}}$$

Question 4 [10 points]:

(a) [1] Convert $-330°$ to radians.

$$-330° \left(\frac{\pi \text{ radians}}{180°} \right) = \boxed{\frac{-11\pi}{6}}$$

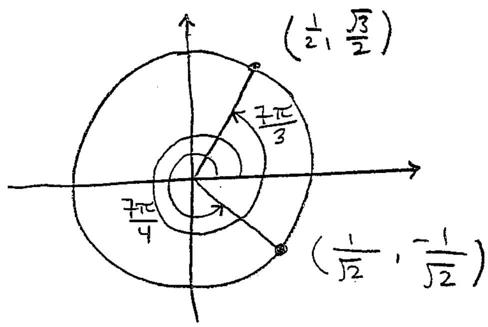
(b) [2] Determine the exact value of $\cos\left(\frac{49\pi}{6}\right)$.

$$\cos\left(\frac{49\pi}{6}\right) = \cos\left(\frac{48\pi}{6} + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

(c) [3] Determine the exact value of $\cos\left(\frac{7\pi}{4}\right)\csc\left(\frac{7\pi}{3}\right)$.

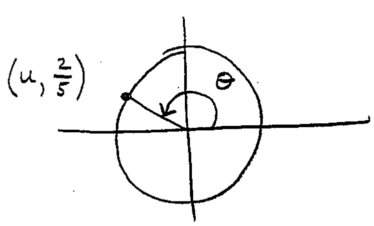
$$\cos\left(\frac{7\pi}{4}\right)\csc\left(\frac{7\pi}{3}\right)$$

$$= \frac{\cos\left(\frac{7\pi}{4}\right)}{\sin\left(\frac{7\pi}{3}\right)}$$



$$= \frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{2}{\sqrt{6}} = \boxed{\frac{\sqrt{6}}{3}}$$

(d) [4] Suppose $\sin\theta = 2/5$ where $\frac{\pi}{2} < \theta < \pi$. Determine $\tan\theta$.



$$\therefore \cos\theta = u = -\sqrt{1 - \left(\frac{2}{5}\right)^2}$$

$$= -\sqrt{1 - \frac{4}{25}}$$

$$= -\sqrt{\frac{21}{25}}$$

$$= -\frac{\sqrt{21}}{5}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\left(2/5\right)}{\left(\frac{-\sqrt{21}}{5}\right)}$$

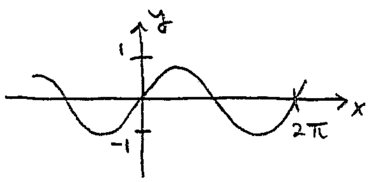
$$= -\frac{2}{\sqrt{21}}$$

$$= \boxed{\frac{-2\sqrt{21}}{21}}$$

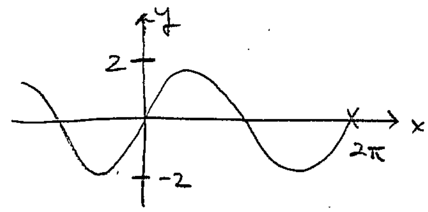
Question 5 [10 points]:

(a) [7] Carefully sketch the graph of $y = 2\sin(3x - \pi) - 2$. $= 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] - 2$

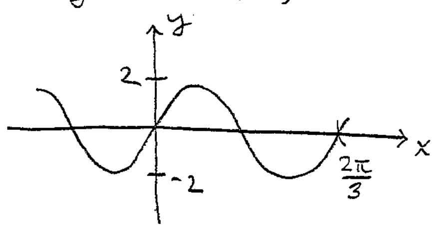
① $y = \sin x$



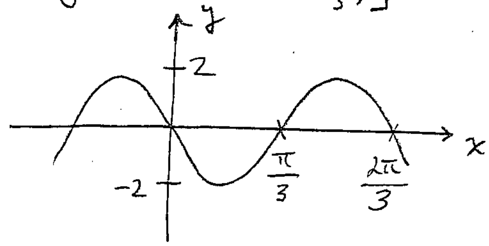
② $y = 2\sin x$



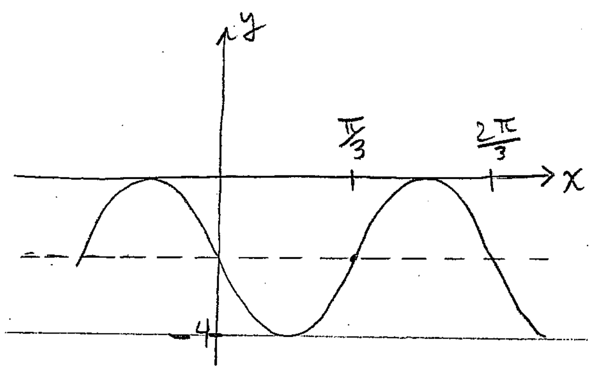
③ $y = 2\sin(3x)$



④ $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right]$



⑤ $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] - 2$



(b) [3] State the period, amplitude and phase-shift of the function sketched in part (a).

period: $\frac{2\pi}{3}$

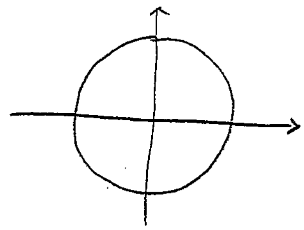
phase-shift: $\frac{\pi}{3}$

amplitude: 2.

Question 6 [10 points]: Find all solutions in the interval $[0, 2\pi)$:

(a) [5] $\sin(2\theta)\sin(\theta) = \cos(\theta)$

$$2\sin\theta\cos\theta\sin\theta = \cos\theta$$



$$2\sin^2\theta\cos\theta - \cos\theta = 0$$

$$(2\sin^2\theta - 1)\cos\theta = 0$$

$$\therefore 2\sin^2\theta - 1 = 0, \quad \cos\theta = 0$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}, \quad \cos\theta = 0$$

$$\therefore \boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2},}$$

(b) [5] $4(1 + \sin(\theta)) = \cos^2(\theta)$

$$4 + 4\sin(\theta) = 1 - \sin^2\theta$$

$$\sin^2\theta + 4\sin\theta + 3 = 0$$

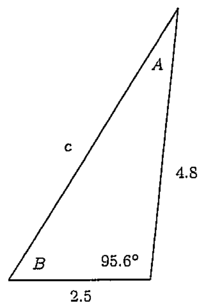
$$(\sin\theta + 3)(\sin\theta + 1) = 0$$

$$\sin\theta + 3 = 0, \quad \sin\theta + 1 = 0$$

$$\underbrace{\sin\theta = -3}_{\text{no solutions}}, \quad \underbrace{\sin\theta = -1}_{\boxed{\theta = \frac{3\pi}{2}.}}$$

Question 7 [10 points]:

(a) [5] Solve for all missing sides and angles in the following triangle. Round final answers to one decimal place.



$c^2 = (2.5)^2 + (4.8)^2 - 2(2.5)(4.8)\cos(95.6)$

$\therefore C \approx 5.62$

$$\frac{\sin(95.6)}{C} = \frac{\sin(B)}{4.8}$$

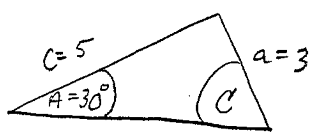
$$\therefore B = \sin^{-1}\left[\frac{4.8\sin(95.6)}{5.62}\right] \approx 58.14^\circ$$

$\therefore A = 180 - 58.14 - 95.6 \approx 26.26$

$\therefore C \approx 5.6$
$A \approx 26.3$
$B \approx 58.1$

(b) [5] A triangle has angle $A = 30°$, side $c = 5$ and side $a = 3$. Determine the two possible values for angle C.

Case 1:

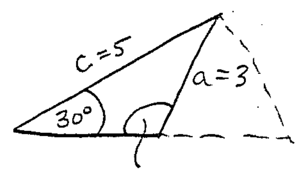


$$\therefore \frac{\sin(30)}{3} = \frac{\sin(C)}{5}$$

$$\therefore C = \sin^{-1}\left[\frac{5\sin(30)}{3}\right]$$

$$\approx \boxed{56.4^\circ}$$

Case 2:

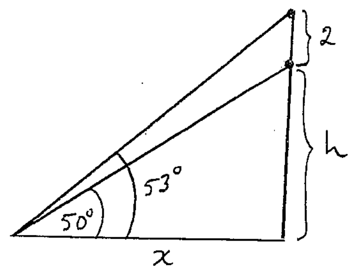


$C \approx 180 - 56.4$

$= \boxed{123.6^\circ}$

Question 8 [10 points]:

(a) [5] An observer on the ground is looking up at a 2 m tall statue mounted atop a building. The angle of elevation to the bottom of the statue is 50°, while that to the top of the statue is 53°. How tall is the building? Round your final answer to one decimal place.



$$\tan(50) = \frac{h}{x} \Rightarrow x = \frac{h}{\tan(50)}$$

$$\tan(53) = \frac{h+2}{x} \qquad \therefore \tan(53) = \frac{h+2}{\frac{h}{\tan(50)}}$$

$$\therefore h\,\frac{\tan(53)}{\tan(50)} = h+2$$

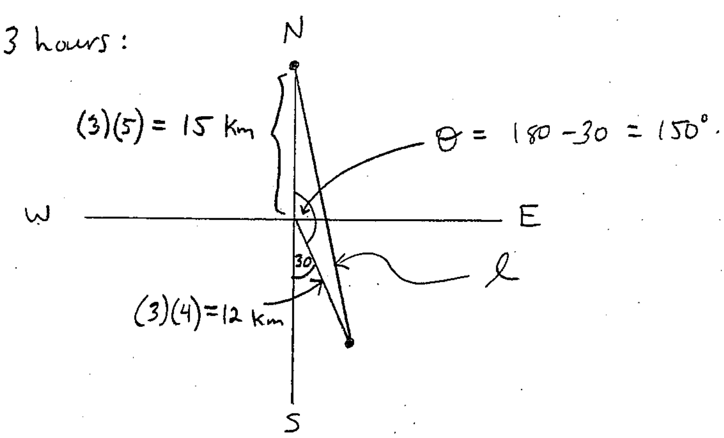
$$h\left[\frac{\tan(53)}{\tan(50)} - 1\right] = 2$$

$$h = \frac{2}{\left[\frac{\tan(53)}{\tan(50)} - 1\right]} \approx 17.6\text{ m}$$

∴ Building is approximately 17.6 m tall.

(b) [5] Two people leave from the same point: one walks north at 5 km/h, the other walks at 4 km/h at a bearing of $S30°E$. What is the distance between the two people three hours later? Again, round your final answer to one decimal place.

At $t = 3$ hours:



∴ distance ℓ is given by

$$\ell^2 = (12)^2 + (15)^2 - 2(12)(15)\cos(150°)$$

$$\boxed{\therefore \ell \approx 26.1\text{ km.}}$$

Question 9 [10 points]: For this question use

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix},\ E = \begin{bmatrix} 2 & 1 \\ 8 & -1 \\ 6 & 5 \end{bmatrix},\ F = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}.$$

(a) [2] Compute $2A - 5C$.

$$2A - 5C = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 6 & -1 \end{bmatrix}$$

(b) [4] Compute $D(F - E)$.

$$F - E = \begin{bmatrix} 3 & -2 \\ -12 & 1 \\ -4 & -2 \end{bmatrix}$$

$$D(F-E) = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -12 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -39 & -2 \\ -8 & -18 \end{bmatrix}$$

(c) [4] Compute A^{-1}.

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$R_1 = \tfrac{1}{2} r_1 : \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]$$

$$R_2 = r_2 - 3r_1 : \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right]$$

$$\to R_2 = 2r_2 : \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

$$R_1 = r_1 - \tfrac{1}{2} r_2 : \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

$$\therefore\ A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Question 10 [10 points]: Solve the following system of equations using matrix reduction (no credit will be given for using any other method):

$$\begin{aligned} x + y + z &= 1 \\ -2x + y + z &= -2 \\ 3x + 6y + 6z &= 5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2 & 1 & 1 & -2 \\ 3 & 6 & 6 & 5 \end{array}\right]$$

$R_2 = r_2 + 2r_1$:

$R_3 = r_3 - 3r_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 2 \end{array}\right]$$

$R_2 = \frac{1}{3} \cdot r_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \end{array}\right]$$

$R_3 = r_3 - 3r_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

no solutions!
(system is inconsistent)

Question 11 [10 points]:

(a) [3] Find a_{20} in the arithmetic sequence $5, 9/2, 4, 7/2, \ldots$.

$a = 5$

$d = \frac{9}{2} - 5 = -\frac{1}{2}$

$\therefore\ a_n = a + (n-1)d$

$a_{20} = 5 + (20-1)\left(-\frac{1}{2}\right)$

$= 5 - \frac{19}{2}$

$= \boxed{-\frac{9}{2}}$

(b) [3] If $a_5 = 2/3$ and $a_7 = 10/3$ in a geometric sequence, what must be the common ratio r if it is a positive number?

$a_7 = a_5 \cdot r^2$

$\therefore\ r^2 = \frac{a_7}{a_5} = \frac{\left(\frac{10}{3}\right)}{\left(\frac{2}{3}\right)} = 5$

$\therefore\ \boxed{r = \sqrt{5}.}$

(c) [4] The arithmetic series $2 + \cdots + 18 = 500$. How many terms are in the series?

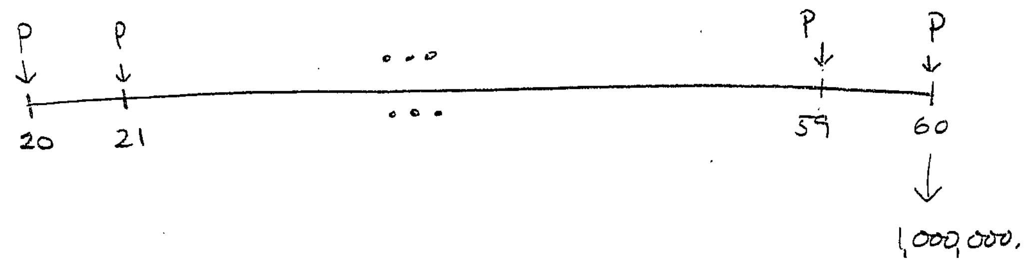
$S_n = n\left[\frac{a_1 + a_n}{2}\right]$

$\therefore\ n = \frac{2S_n}{a_1 + a_n}$

$= \frac{(2)(500)}{2 + 18}$

$= \boxed{50}$

Question 12 [10 points]: A person would like to accumulate \$1,000,000 by his 60^{th} birthday so he can retire early. The plan is to make equal deposits into a fund on each birthday beginning with the 20^{th} and ending with the final payment made on the 60^{th}. If all deposits are invested in a fund paying 5% interest compounded annually, how large must the equal annual deposits be in order to achieve the \$1,000,000 goal? Recall that P dollars invested at $i\%$ interest compounded annually accumulates to an amount $A = P\left(1 + \frac{i}{100}\right)^n$ by the end of n years.



$$\therefore \quad P + P\left(1+\frac{5}{100}\right) + P\left(1+\frac{5}{100}\right)^2 + \cdots + P\left(1+\frac{5}{100}\right)^{40} = 1{,}000{,}000$$

$$P\Big[\underbrace{1 + 1.05 + (1.05)^2 + \cdots + (1.05)^{40}}_{\text{geometric, } a=1,\ r=1.05}\Big] = 1{,}000{,}000$$

$$\therefore \quad P\left[\frac{1-(1.05)^{41}}{1-1.05}\right] = 1{,}000{,}000$$

$$\therefore \quad P = \frac{(1\,000\,000)(1-1.05)}{1-(1.05)^{41}} \simeq \$7822.29$$

$$\boxed{\therefore \text{ annual payments should be } \$7822.29.}$$