Question 1 [10 points]:

(a) [5] Determine $f^{-1}(x)$ if $f(x) = \dfrac{3x+1}{2x-1}$.

$$y = \frac{3x+1}{2x-1}$$

$$x = \frac{3y+1}{2y-1}$$

$$2xy - x = 3y+1$$

$$2xy - 3y = x+1$$

$$y(2x-3) = x+1$$

$$y = \frac{x+1}{2x-3}$$

$$\boxed{\therefore\ f^{-1}(x) = \frac{x+1}{2x-3}}$$

(b) [3] Express as a single logarithm: $\log_5(x+3) - \log_5(3x+1) - 3\log_5 2$

$$= \log_5(x+3) - \log_5(3x+1) - \log_5 2^3$$

$$\boxed{= \log_5 \frac{(x+3)}{(3x+1)\cdot 8}}$$

(c) [2] Calculate $\log_\pi 2$ (round your answer to two decimal places).

$$\log_\pi 2 = \frac{\ln 2}{\ln \pi} \approx \boxed{0.61}$$

Question 2 [10 points]: Solve the following equations for x:

(a) [5] $\dfrac{e^{x^2}}{e^6} = e^x$

$$e^{x^2} = e^{x+6}$$

$$x^2 = x+6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$\boxed{x = 3, \quad x = -2}$$

(b) [5] $\log_2(x+2) + \log_2(x-1) = 2$

$$\log_2[(x+2)(x-1)] = 2$$

$$(x+2)(x-1) = 4$$

$$x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

$$x = 2, \quad x = -3$$

$\boxed{\therefore\ x = 2 \text{ is the only solution.}}$

Check: $x=2$: $\log_2(2+2) + \log_2(2-1) \;\big|\; 2$

$\log_2(4) + \cancel{\log_2(1)}^{0} \;\big|\; 2$

$2 = 2$ ✓

$x=-3$: $\log_2(-3+2) + \log_2(-3-1) \;\big|\; 2$

$\underbrace{\log_2(-1) + \log_2(-4)}_{\text{not defined.}} \;\big|\; 2$

Question 3 [10 points]:

(a) [5] A 10 g sample of radioactive material reduces to 6 g after 25 days. What is the half life of the material? Round your answer to two decimal places. (Recall that the amount A of radioactive material present at time t is given by $A(t) = A_0e^{kt}$ where A_0 is the original amount of radioactive material and k is a negative constant.)

$$6 = e^{k \cdot 25}$$

$$\frac{6}{10} = e^{25k}$$

$$\therefore k = \frac{1}{25}\ln\left(\frac{6}{10}\right)$$

Now solve $\frac{1}{2}\cancel{A_0} = \cancel{A_0}e^{kt}$

$$kt = \ln(1/2)$$

$$t = \frac{\ln(1/2)}{k} = \frac{\ln(1/2)}{\frac{1}{25}\ln\left(\frac{6}{10}\right)}$$

$$\boxed{t \approx 33.92 \text{ days.}}$$

(b) [5] A population grows according to the model $P(t) = P_0e^{kt}$, where k is a positive constant, t is time in years, and $t = 0$ corresponds to the present. If the population in five years time is 5000 and reaches 7000 after a further two years, what is the current population size? Round your answer to two decimal places.

$$5000 = P_0 e^{5k}$$

$$7000 = P_0 e^{7k}$$

$$\therefore \frac{7000}{5000} = \frac{P_0 e^{7k}}{P_0 e^{5k}}$$

$$\frac{7}{5} = e^{2k}$$

$$\therefore k = \frac{1}{2}\ln\left(\frac{7}{5}\right)$$

Now solve $5000 = P_0 e^{5k}$ for P_0 :

$$P_0 = \frac{5000}{e^{5k}} = \frac{5000}{e^{5 \cdot \frac{1}{2}\ln\left(\frac{7}{5}\right)}} \approx \boxed{2156. \text{ individuals.}}$$

Question 4 [10 points]:

(a) [2] Convert $31\pi/9$ to degrees.

$$\left(\frac{31\cancel{\pi}}{\cancel{9}}\right)\left(\frac{\overset{20}{\cancel{180}}}{\cancel{\pi}}\right) = \boxed{620^\circ}$$

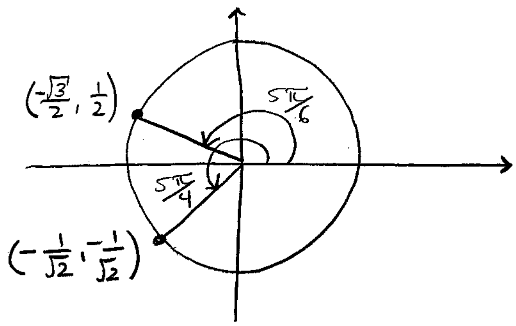
(b) [3] Determine the exact value of $\sin\left(\frac{5\pi}{6}\right)\sec\left(\frac{5\pi}{4}\right)$.

$$\sin\left(\frac{5\pi}{6}\right)\sec\left(\frac{5\pi}{4}\right)$$

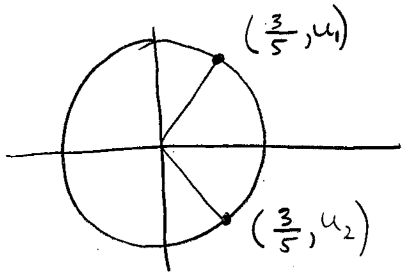
$$= \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{4}\right)}$$

$$= \frac{\left(\frac{1}{2}\right)}{\left(\frac{-1}{\sqrt{2}}\right)}$$

$$= \boxed{-\frac{\sqrt{2}}{2}}$$



(d) [4] If $\cos\theta = 3/5$ determine all possible values of $\sin\theta$.



$$u^2 + \left(\frac{3}{5}\right)^2 = 1$$

$$\therefore\ u = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$u = \pm\sqrt{\frac{25-9}{25}}$$

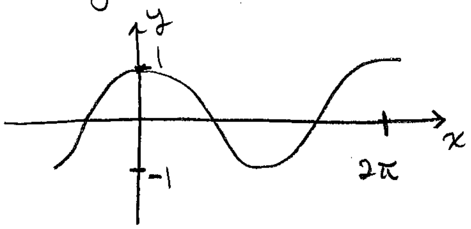
$$u = \pm\frac{4}{5}$$

$$\boxed{\therefore\ \sin\theta = \frac{4}{5} \text{ or } -\frac{4}{5}.}$$

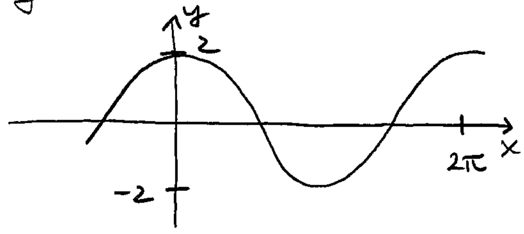
Question 5 [10 points]:

(a) [7] Carefully sketch the graph of $y = 2\cos(4x - \pi) + 1$. $= 2\cos\left[4\left(x - \frac{\pi}{4}\right)\right] + 1$

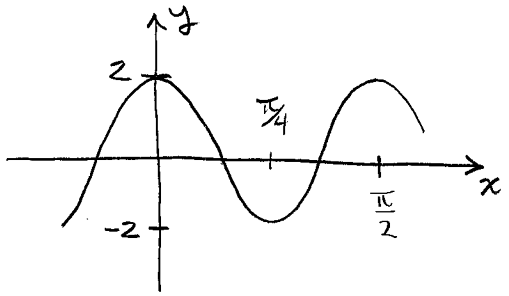
① $y = \cos x$



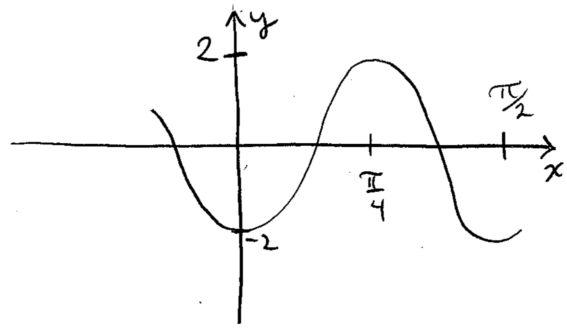
② $y = 2\cos x$



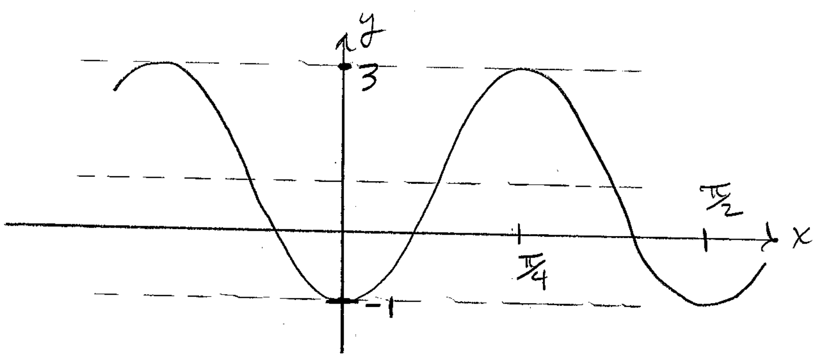
③ $y = 2\cos(4x)$



④ $y = 2\cos\left[4\left(x - \frac{\pi}{4}\right)\right]$



⑤ $y = 2\cos\left[4\left(x - \frac{\pi}{4}\right)\right] + 1$



(b) [3] State the period, amplitude and phase-shift of the function sketched in part (a).

period : $\frac{2\pi}{4} = \frac{\pi}{2}$

amplitude : $|2| = 2$

phase-shift : $\frac{\pi}{4}$.

Question 6 [10 points]: Find all solutions in the interval $[0, 2\pi)$:

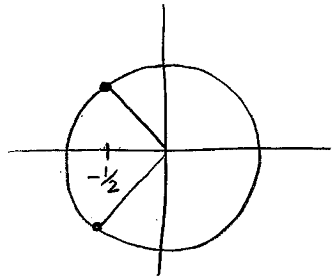
(a) [5] $2\cos^2\theta - 3\cos\theta = 2$

$$2\cos^2\theta - 3\cos\theta - 2 = 0$$

$$2\cos^2\theta - 4\cos\theta + \cos\theta - 2 = 0$$

$$2\cos\theta\left[\cos\theta - 2\right] + \cos\theta - 2 = 0$$

$$(\cos\theta - 2)(2\cos\theta + 1) = 0$$

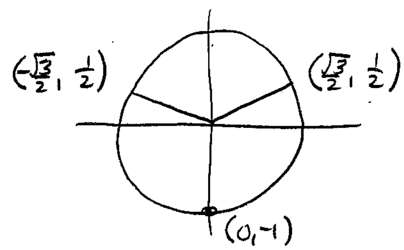


$\therefore$ $\underbrace{\cos\theta = 2}_{\text{no solutions}}$, $2\cos\theta + 1 = 0$

$\cos\theta = \frac{-1}{2}$

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$$

(b) [5] $\cos(2\theta) = \sin(\theta)$



$$\cos^2\theta - \sin^2\theta = \sin\theta$$

$$1 - \sin^2\theta - \sin^2\theta = \sin\theta$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$\sin\theta = \frac{-1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$$

$$= \frac{-1 \pm 3}{4}$$

$$= \frac{1}{2}, -1$$

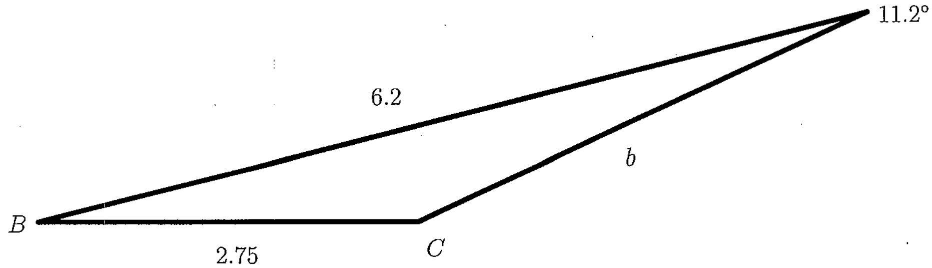
$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

$\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$.

$$\boxed{\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.}$$

Question 7 [10 points]:

(a) [5] Solve for all missing sides and angles in the following triangle. The figure is to scale. Round final answers to one decimal place.



$$\frac{\sin(11.2)}{2.75} = \frac{\sin(C)}{6.2}$$

$$\therefore\ C = \underbrace{180 -}_{} \sin^{-1}\left[\frac{(6.2)\sin(11.2)}{2.75}\right]$$

NOTE! (since C is obtuse)

$$C \approx 154.029^\circ$$

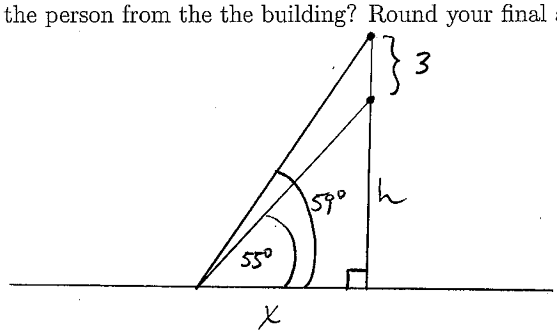
$$\therefore\ B = 180 - 11.2 - C \approx 14.771^\circ$$

$$\frac{\sin(11.2)}{2.75} = \frac{\sin(14.771)}{b}$$

$$\therefore\ b = \frac{(2.75)\sin(14.771)}{\sin(11.2)} \approx 3.610$$

$$\boxed{\begin{aligned} \therefore\ C &\approx 154.0^\circ \\ B &\approx 14.8^\circ \\ b &\approx 3.6 \end{aligned}}$$

(b) [5] An observer on the ground is looking up at a 3 m tall antenna mounted atop a building. The angle of elevation to the bottom of the antenna is 55°, while that to the top of the antenna is 59°. How far is the person from the the building? Round your final answer to one decimal place.



$$\tan(55) = \frac{h}{x} \Rightarrow h = x\tan(55)$$

$$\tan(59) = \frac{h+3}{x} \Rightarrow \tan(59) = \frac{x\tan(55)+3}{x}$$

$$x\tan(59) - x\tan(55) = 3$$

$$\therefore\ x = \frac{3}{\tan(59) - \tan(55)} \approx 12.7\ \text{m}$$

$\boxed{\therefore \text{ The observer is } 12.7 \text{ m from the building.}}$

Question 8 [10 points]: Let

$$A = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

(a) [3] Compute $(4A - 3B)C$.

$$4A - 3B = \begin{bmatrix} -8 & 12 \\ 4 & -4 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} -8 & 15 \\ -2 & 5 \end{bmatrix}$$

$$(4A-3B)C = \begin{bmatrix} -8 & 15 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 68 & -16 & 51 \\ 22 & -4 & 19 \end{bmatrix}.$$

(b) [2] If there is a matrix D such that the product CDA is defined, what must be the size of D?.

$$\begin{array}{ccc} C & D & A \\ 2\times 3 & \downarrow & 2\times 2 \end{array}$$

$\therefore 3 \times 2$

$\therefore$ D must be of size 3×2.

(c) [5] Determine A^{-1} .

$$\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$: $\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ -2 & 3 & 1 & 0 \end{array}\right]$

$R_2 = r_2 + 2r_1$: $\left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$R_1 = r_1 + r_2$: $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$

$$\therefore A^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Question 9 [10 points]: Solve the following system of equations **using matrix reduction** (no credit will be given for using any other method). Clearly state your conclusion:

$$x + y + z = 0$$
$$x - 2y + 2z = 4$$
$$x + 2y - z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right]$$

$R_2 = r_2 - r_1$:
$R_3 = r_3 - r_1$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

$R_2 \leftrightarrow R_3$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 1 & 4 \end{array}\right]$$

$R_3 = r_3 + 3r_2$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & 10 \end{array}\right]$$

$R_3 = \frac{-1}{5} r_3$:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\therefore z = -2$

$y = 2 + 2z = 2 + 2(-2) = -2$

$x = 0 - y - z = 0 - (-2) - (-2) = 4$

$\therefore \{x = 4, y = -2, z = -2\}$.

Question 10 [10 points]:

(a) [2] Determine a_{11} in the geometric sequence $\pi, -\pi/\sqrt{2}, \pi/2, \ldots$.

$$a = \pi$$

$$r = \frac{\left(-\frac{\pi}{\sqrt{2}}\right)}{\pi} = -\frac{1}{\sqrt{2}}$$

$$\therefore\ a_{11} = ar^{10} = \pi\left(-\frac{1}{\sqrt{2}}\right)^{10} = \pi \frac{1}{(2^{\frac{1}{2}})^{10}} = \frac{\pi}{2^5}$$

$$\therefore\ \boxed{a_{11} = \frac{\pi}{32},}$$

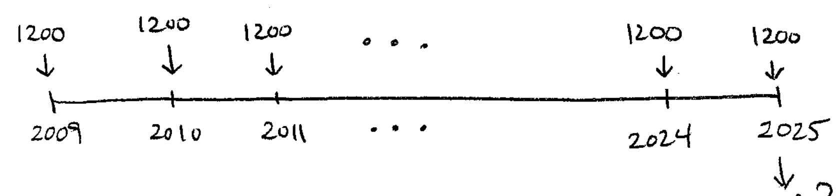
(b) [3] Compute the sum of the arithmetic series $7 + 11/2 + \cdots + a_{23}$.

$$a = 7, \quad d = \frac{11}{2} - 7 = \frac{11}{2} - \frac{14}{2} = -\frac{3}{2}, \quad n = 23.$$

$$\therefore\ S_{23} = \frac{23\left[(2)(7) + (23-1)\left(-\frac{3}{2}\right)\right]}{2}$$

$$= \boxed{\frac{-437}{2}}$$

(c) [5] \$1200 is deposited on January 1 of each year into a fund which pays interest at 6.5% compounded annually. The first deposit is made on January 1 2009. What is the accumulated value of the fund just after the last deposit is made on January 1 2025? Round your answer to the nearest dollar. Recall that P dollars invested at $i\%$ interest compounded annually accumulates to an amount $A = P\left(1 + \frac{i}{100}\right)^n$ by the end of n years.



$$A = 1200 + 1200\left(1 + \frac{6.5}{100}\right) + 1200\left(1 + \frac{6.5}{100}\right)^2 + \cdots + 1200\left(1 + \frac{6.5}{100}\right)^{16}$$

$$= 1200\Big[\underbrace{1 + (1.065) + (1.065)^2 + \cdots + (1.065)^{16}}_{\text{geometric : } a=1,\ r=1.05,\ n=17}\Big]$$

$$\therefore\ A = 1200\left[\frac{1-(1.065)^{17}}{1-1.065}\right] \approx \boxed{\$35,392}$$