①

$$\log(8x) - \log(1+\sqrt{x}) = 2$$

$$\log\left(\frac{8x}{1+\sqrt{x}}\right) = 2$$

$$\therefore \frac{8x}{1+\sqrt{x}} = 100$$

$$8x = 100 + 100\sqrt{x}$$

$$8x - 100\sqrt{x} - 100 = 0$$

$$2x - 25\sqrt{x} - 25 = 0$$

let $t = \sqrt{x}$, so equation is

$$2t^2 - 25t - 25 = 0$$

$$t = \frac{25 \pm \sqrt{25^2 - 4(2)(-25)}}{2(2)}$$

$$= \frac{25 \pm 5\sqrt{33}}{4}$$

$$= \frac{25 + 5\sqrt{33}}{4}, \quad \cancel{\frac{25 - 5\sqrt{33}}{4}}$$

since $t > 0$

$$\therefore x = t^2 = \left(\frac{25+5\sqrt{33}}{4}\right)^2$$

Now check in original equation to confirm this is indeed the solution

---

②

$$430 = 250e^{10k}$$

$$\therefore k = \frac{1}{10}\ln\left(\frac{430}{250}\right)$$

$$\therefore N = 250e^{\left[\frac{1}{10}\ln\left(\frac{430}{250}\right)\right]t}$$

Now solve $250e^{\left[\frac{1}{10}\ln\left(\frac{430}{250}\right)\right]t} = 500$

$$\therefore \frac{1}{10}\ln\left(\frac{430}{250}\right)t = \ln(2)$$

$$\therefore t = \frac{10\ln(2)}{\ln\left(\frac{430}{250}\right)} \doteq \boxed{12.8 \text{ hrs.}}$$

---

③

Solve $2000e^{0.06t} = 2400e^{0.055t}$ for t

$$\therefore \frac{e^{0.06t}}{e^{0.055t}} = \frac{2400}{2000}$$

$$e^{0.005t} = \frac{6}{5}$$

$$\therefore t = \frac{1}{0.005}\ln\left(\frac{6}{5}\right) \doteq \boxed{36.5 \text{ yrs.}}$$

④ $2\sin^2\theta - 3\sin\theta + 1 = 0$.

Let $x = \sin\theta$

$\therefore$ Solve $2x^2 - 3x + 1 = 0$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{3 \pm 1}{4}$$

$$= 1, \frac{1}{2}$$

$\therefore$ $\sin\theta = 1$, $\sin\theta = \frac{1}{2}$

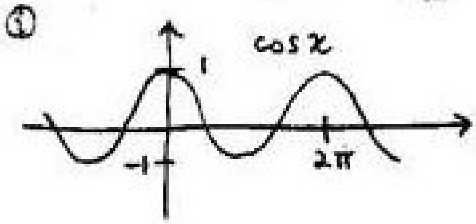
$\therefore$ $\boxed{\theta = \frac{\pi}{2}}$, $\boxed{\theta = \frac{\pi}{6}}$

---

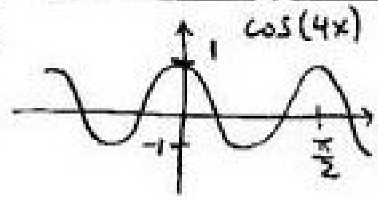
⑤ $y = -3\cos[4x - \pi] + 2$

$= -3\cos\left[4\left(x - \frac{\pi}{4}\right)\right] + 2$

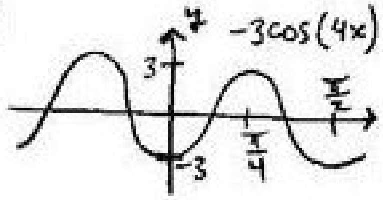
(i) $\cos x$



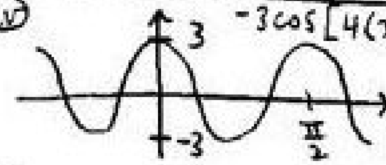
(ii) $\cos(4x)$



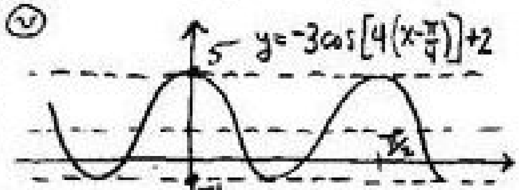
(iii) $-3\cos(4x)$



(iv) $-3\cos\left[4\left(x - \frac{\pi}{4}\right)\right]$



(v) $y = -3\cos\left[4\left(x - \frac{\pi}{4}\right)\right] + 2$



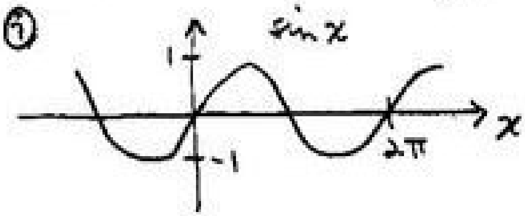
Period $\frac{\pi}{2}$, amplitude 3, phase-shift $\frac{\pi}{4}$

---

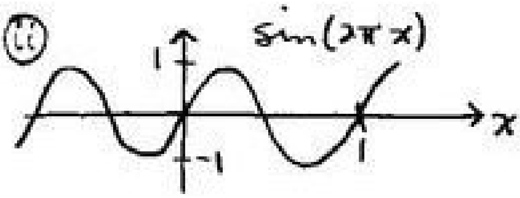
⑥ $y = \frac{1}{2}\sin(2\pi x + \pi) - 1$

$y = \frac{1}{2}\sin\left[2\pi\left(x + \frac{1}{2}\right)\right] - 1$

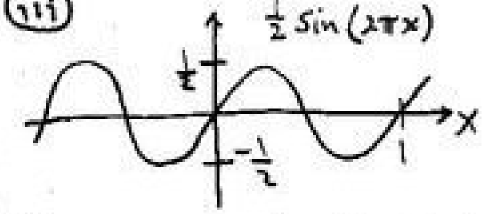
(i) $\sin x$



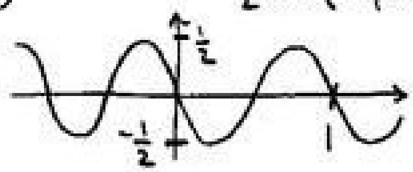
(ii) $\sin(2\pi x)$



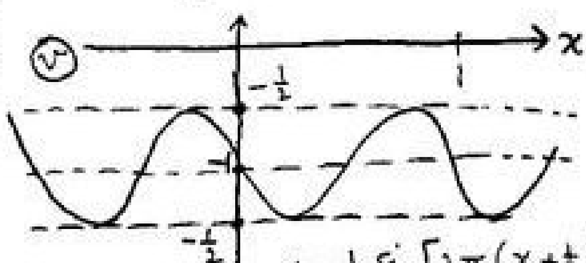
(iii) $\frac{1}{2}\sin(2\pi x)$



(iv) $\frac{1}{2}\sin\left(2\pi\left(x + \frac{1}{2}\right)\right)$



(v)



$y = \frac{1}{2}\sin\left[2\pi\left(x + \frac{1}{2}\right)\right] - 1$

Period 1, amplitude $\frac{1}{2}$, phase-shift $-\frac{1}{2}$

⑦ (a) $\cos\theta = \sqrt{\frac{11}{13}}$

(b) $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\sqrt{\frac{2}{13}}} = -\sqrt{\frac{13}{2}}$

(c) $\sin(\theta+\pi) = \sin\theta\cos\pi + \underbrace{\cancel{\cos\theta\sin\pi}}_{0}$

$= -\sqrt{\frac{2}{13}}\cdot(-1) = \sqrt{\frac{2}{13}}$.

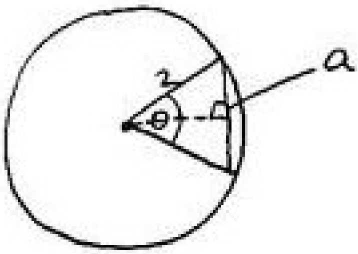
(d) $\cos\left(\theta+\frac{\pi}{2}\right) = \cos\theta\underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0} - \sin\theta\overbrace{\cancel{\sin\left(\frac{\pi}{2}\right)}}^{1} = -\left(-\sqrt{\frac{2}{13}}\right) = \sqrt{\frac{2}{13}}$.

$\sin\left(\theta+\frac{\pi}{2}\right) = \sin\theta\underbrace{\cancel{\cos\left(\frac{\pi}{2}\right)}}_{0} + \cos\theta\overbrace{\cancel{\sin\left(\frac{\pi}{2}\right)}}^{1} = \sqrt{\frac{11}{13}}$

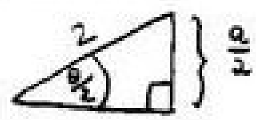
$\therefore$ coordinates are $\left(\sqrt{\frac{2}{13}}, \sqrt{\frac{11}{13}}\right)$.

---

⑧



Bisect θ to form triangle

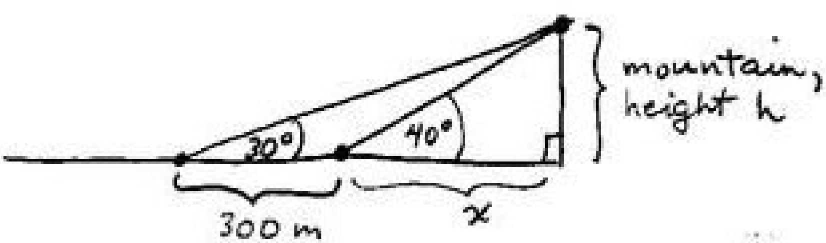


$\therefore \sin\left(\frac{\theta}{2}\right) = \frac{a/2}{2} = \frac{a}{4}$

$\therefore \boxed{a = 4\sin\left(\frac{\theta}{2}\right)}$

---

⑨



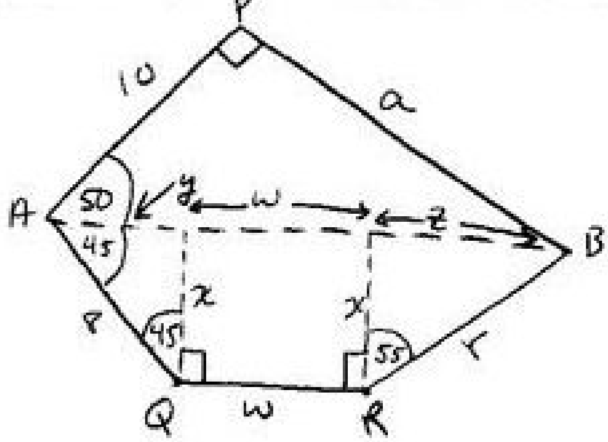
$\left.\begin{aligned} \tan(40°) &= \frac{h}{x} \\ \tan(30°) &= \frac{h}{300+x} \end{aligned}\right\}$ $\therefore \left.\begin{aligned} x &= \frac{h}{\tan(40)} \\ x &= \frac{h}{\tan(30)} - 300 \end{aligned}\right\}$ $\therefore \frac{h}{\tan(40)} = \frac{h}{\tan(30)} - 300$

$h\left[\frac{1}{\tan(40)} - \frac{1}{\tan(30)}\right] = -300$

$\therefore h = \frac{-300}{\left[\frac{1}{\tan(40)} - \frac{1}{\tan(30)}\right]}$

$\boxed{h \doteq 555.2 \text{ m}}$

(10)



$\tan(50) = \frac{a}{10}$

$\therefore a = 10\tan(50)$

$\therefore$ Route APB $= 10 + 10\tan(50)$

$\doteq 21.9$

$\sin(45) = \frac{x}{8}$; $\therefore x = 8\sin(45) = \frac{8}{\sqrt{2}}$.

$\cos(45) = \frac{y}{8}$; $\therefore y = 8\cos(45) = \frac{8}{\sqrt{2}}$.

$\tan(55) = \frac{z}{x}$; $\therefore z = x\tan(55)$

$= \frac{8}{\sqrt{2}}\tan(55)$

$AB = \sqrt{10^2 + a^2} = \sqrt{100 + [10\tan(50)]^2}$

and $w = AB - y - z$

$= \sqrt{100 + 100(\tan(50))^2} - \frac{8}{\sqrt{2}} - \frac{8}{\sqrt{2}}\tan(55)$

Finally, $r = \sqrt{x^2 + z^2}$

$= \sqrt{\frac{64}{2} + \frac{64}{2}(\tan(55))^2}$.

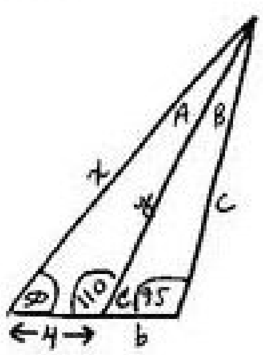
$\therefore$ Route $AQRB = 8 + w + r$

$\doteq 19.7$

$\therefore$ Bottom route (AQRB) is shorter!

---

(11)



$A = 180 - 110 - 50 = \boxed{20^\circ}$

$\left.\frac{\sin 50}{y} = \frac{\sin A}{4}\right\} \therefore y = \frac{4\sin(50)}{\sin(20)} \doteq \boxed{9.0}$

$x^2 = 4^2 + y^2 - 2(4)(y)\cos(110)$

$\therefore x = \sqrt{16 + y^2 - 8y\cos(110)} \doteq \boxed{11.0}$

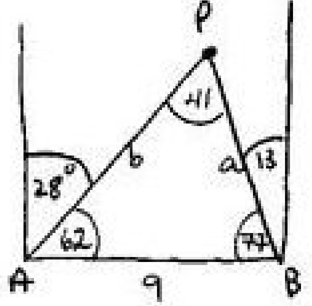
$C = 180 - 110 = \boxed{70^\circ}$

$\therefore B = 180 - C - 95 = 180 - 70 - 95 = \boxed{15^\circ}$

$\left.\frac{\sin(95)}{y} = \frac{\sin B}{b}\right\} \therefore b = \frac{y\sin(15)}{\sin(95)} \doteq \boxed{2.3}$

$\left.\frac{\sin C}{c} = \frac{\sin 95}{y}\right\} \therefore c = \frac{y\sin C}{\sin 95} \doteq \boxed{8.5}$

(12)



$$\frac{\sin(62)}{a} = \frac{\sin(41)}{9}$$

$$\therefore a = \frac{9\sin(62)}{\sin(41)} \doteq 12.1 \text{ nautical miles}$$

$$\frac{\sin(77)}{b} = \frac{\sin(41)}{9}$$

$$\therefore b = \frac{9\sin(77)}{\sin(41)} \doteq 13.4 \text{ nautical miles.}$$

$\therefore$ Time to reach P from A $= \frac{b}{40} \doteq 0.33$ hr.

Time to reach P from B $= \frac{a}{35} \doteq 0.35$ hr.

$\therefore$ Station A should respond to reach P as quickly as possible.

---

(13)

$$\tan\left(\frac{\pi}{8}\right) = \frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}.$$

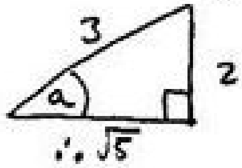
From $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$, we have $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{2}$

From $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, we have $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos(\theta)}{2}$.

$$\therefore \tan^2\left(\frac{\pi}{8}\right) = \frac{\sin^2\left(\frac{\pi}{8}\right)}{\cos^2\left(\frac{\pi}{8}\right)} = \frac{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}}{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

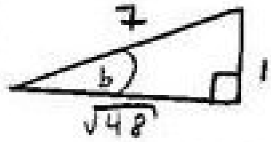
$$\boxed{\therefore \tan\left(\frac{\pi}{8}\right) = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}} \left.\vphantom{\sqrt{\frac{1}{1}}}\right\} \text{positive square root since } \frac{\pi}{8} \text{ in first quadrant.}$$

(14) $\sin a = \frac{2}{3}$:



$\therefore \cos a = \frac{\sqrt{5}}{3}$

$\sin b = \frac{1}{7}$:



$\therefore \cos b = \frac{\sqrt{48}}{7}$

$$\cos(a-b)\cos(a+b) = [\cos a\cos b + \sin a\sin b][\cos a\cos b - \sin a\sin b]$$

$$= \left[\frac{\sqrt{5}}{3}\cdot\frac{\sqrt{48}}{7} + \frac{2}{3}\cdot\frac{1}{7}\right]\left[\frac{\sqrt{5}}{3}\cdot\frac{\sqrt{48}}{7} - \frac{2}{3}\cdot\frac{1}{7}\right]$$

$$= \frac{5\cdot 48}{3\cdot 3\cdot 7\cdot 7} - \frac{2\cdot 2}{3\cdot 3\cdot 7\cdot 7}$$

$$= \boxed{\frac{236}{441}}$$

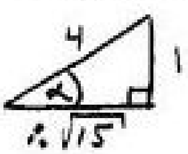
---

(15) $$\sin\left[\sin^{-1}\left(\tfrac{1}{3}\right) + \sin^{-1}\left(\tfrac{1}{4}\right)\right] = \sin\left(\sin^{-1}\left(\tfrac{1}{3}\right)\right)\cos\left(\sin^{-1}\left(\tfrac{1}{4}\right)\right) + \cos\left(\sin^{-1}\left(\tfrac{1}{3}\right)\right)\sin\left(\sin^{-1}\left(\tfrac{1}{4}\right)\right)$$

$$= \frac{1}{3}\cos\left(\sin^{-1}\left(\tfrac{1}{4}\right)\right) + \frac{1}{4}\cos\left(\sin^{-1}\left(\tfrac{1}{3}\right)\right)$$

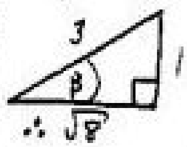
$\alpha = \sin^{-1}\left(\frac{1}{4}\right)$

$\therefore \sin\alpha = \frac{1}{4}$



$\beta = \sin^{-1}\left(\frac{1}{3}\right)$

$\therefore \sin\beta = \frac{1}{3}$



$$= \frac{1}{3}\cos(\alpha) + \frac{1}{4}\cos(\beta)$$

$$= \frac{1}{3}\,\frac{\sqrt{15}}{4} + \frac{1}{4}\,\frac{\sqrt{8}}{3}$$

$$= \boxed{\frac{\sqrt{15}+\sqrt{8}}{12}}$$

---

(16) (a)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R_1+R_3]{(-2)R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 3 & -1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$(-4)R_3+R_1$, $2R_3+R_2$:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -1 & -4 \\ 0 & 1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 7 & -1 & -4 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \quad \therefore \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 7 & -1 & -4 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

(17) (a)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 1 & 0 \\ 2 & 3 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{(-3)R_1+R_2 \\ (-2)R_1+R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -7 & 7 & -3 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \\ 0 & -7 & 7 & -3 & 1 & 0 \end{array}\right]$$

$$(-1)R_2: \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & -7 & 7 & -3 & 1 & 0 \end{array}\right] \xrightarrow{\substack{(-2)R_2+R_1 \\ (7)R_2+R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -3 & 0 & 2 \\ 0 & 1 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 11 & 1 & -7 \end{array}\right]$$

reduced, but not to I, so $\boxed{A^{-1} \text{ does not exist}}$

(b)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow{\substack{(-3)R_1+R_2 \\ (-2)R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & -7 & 7 & 7 \end{array}\right]$$

$$(-1)R_2: \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -7 & 7 & 7 \end{array}\right] \xrightarrow{\substack{(-2)R_2+R_1 \\ (7)R_2+R_3}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = r$
$\therefore y = -1 + z = -1 + r$
$\therefore x = z = r$

$\boxed{\therefore x = r,\ y = r-1,\ z = r \text{ where } r \text{ is any real number}}$

---

(18) (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$

$$A - 2I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$B + 3I = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -2 & 6 \end{bmatrix}$$

$$\therefore (A-2I)(B+3I) = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 7 & -6 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} -11 & 18 \\ 10 & 0 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cc|cc} -11 & 18 & 1 & 0 \\ 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\left(\frac{-1}{11}\right)R_1} \left[\begin{array}{cc|cc} 1 & \frac{-18}{11} & \frac{-1}{11} & 0 \\ 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-10R_1+R_2} \left[\begin{array}{cc|cc} 1 & \frac{-18}{11} & \frac{-1}{11} & 0 \\ 0 & \frac{180}{11} & \frac{10}{11} & 1 \end{array}\right]$$

$$\left(\frac{11}{180}\right)R_2: \left[\begin{array}{cc|cc} 1 & \frac{-18}{11} & \frac{-1}{11} & 0 \\ 0 & 1 & \frac{1}{18} & \frac{11}{180} \end{array}\right] \xrightarrow{\left(\frac{18}{11}\right)R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{10} \\ 0 & 1 & \frac{1}{18} & \frac{11}{180} \end{array}\right]$$

$$\therefore \begin{bmatrix} -11 & 18 \\ 10 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{10} \\ \frac{1}{18} & \frac{11}{180} \end{bmatrix}$$

(19) $\left.\begin{array}{l}(x-d)+x+(x+d)=30 \\ (x-d)\times(x+d)=360\end{array}\right\}$ $3x=30 \rightarrow \therefore x=10$

$(x-d)\times(x+d)=360 \Big\} \therefore (10-d)10(10+d)=360$

$1000-10d^2=360$

$d=\pm\sqrt{\dfrac{1000-360}{10}}$

$d=\pm 8$

$\therefore$ terms are $10-8,\ 10,\ 10+8$,

i.e. $\boxed{2,\ 10,\ 18}$

(note: $d=+8$ and $d=-8$ both give the same three terms.)

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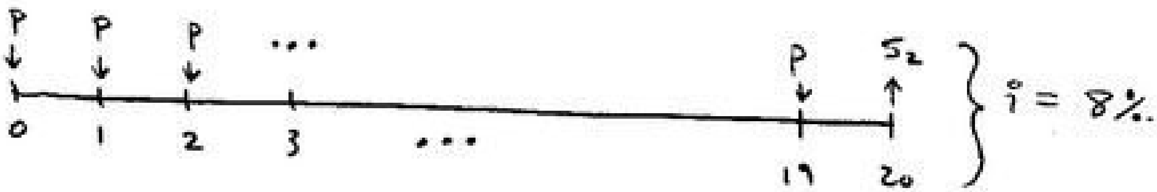
(20)



First investment at 7%:

$P(1.07)+P(1.07)^2+\cdots+P(1.07)^{20}=S_1$ $\Big\}$ geometric, $a=P(1.07)$, $r=1.07$, $n=20$

$\therefore S_1 = P(1.07)\dfrac{(1-1.07^{20})}{1-1.07}$

Second investment at 8%:

$P(1.08)+P(1.08)^2+\cdots+P(1.08)^{20}=S_2$ $\Big\}$ geometric, $a=P(1.08)$, $r=1.08$, $n=20$

$S_2 = P(1.08)\dfrac{(1-1.08^{20})}{1-1.08}$

$\therefore$

$$\frac{S_2}{S_1}=\frac{\dfrac{P(1.08)(1-1.08^{20})}{(1-1.08)}}{\dfrac{P(1.07)(1-1.07^{20})}{(1-1.07)}}=\frac{\cancel{P}(1.08)(1-1.08^{20})(1-1.07)}{(1-1.08)\cancel{P}(1.07)(1-1.07^{20})}$$

$\doteq \boxed{1.13}$