

## 2 Second Derivatives and Shapes of Curves

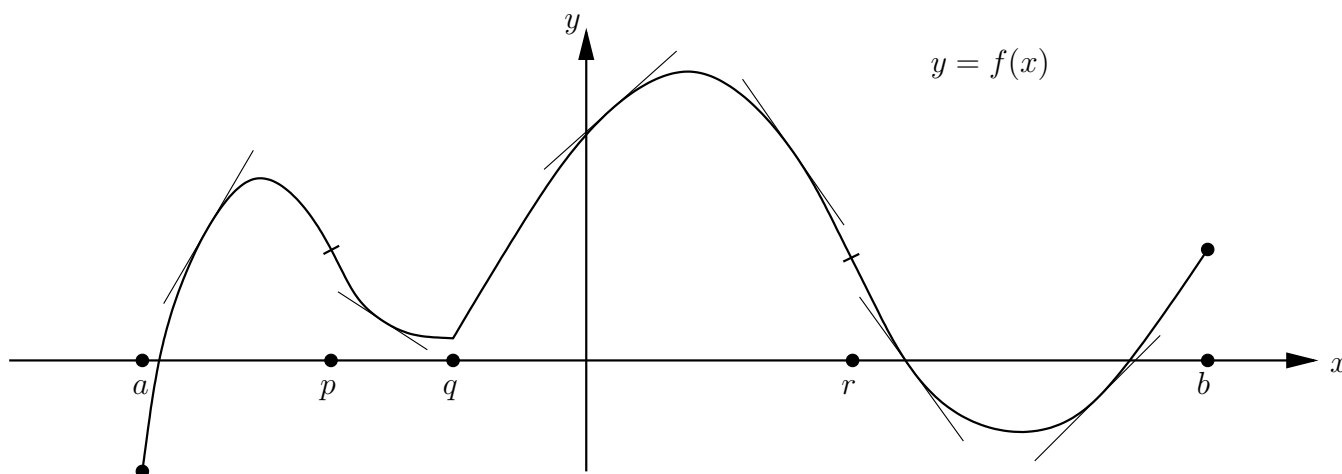
We have seen how  $f'(x)$  gives us information about how a function increases and decreases, in particular:

- (i) If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval.
- (ii) If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval.

Now we examine what the second derivative  $f''(x)$  tells us about  $f$  and its graph. First, recall

$$\begin{aligned} f''(x) &= \frac{d}{dx} [f'(x)] \\ &= \text{the rate of change of } f'(x) \\ &= \text{the slope of tangent lines to } y = f'(x) \end{aligned}$$

To see how this applies to the graph of  $y = f(x)$ , consider a general graph in which some tangent lines have been drawn:



Reading the graph left to right (as always!), notice:

- (i) Slopes of tangent lines decrease over the intervals  $(a, p)$  and  $(q, r)$ . Over these intervals tangent lines lie above the graph, and the graph itself bends downward.
- (ii) Slopes of tangent lines increase over the intervals  $(p, q)$  and  $(r, b)$ . Over these intervals tangent lines lie below the graph, and the graph itself bends upward.
- (iii) The transitions between increasing and decreasing tangent slopes, that is, transitions between bending trends, occur at  $x = p$ ,  $x = q$  and at  $x = r$ .

Now, extend what we learned about first derivatives to second derivatives:

$$\begin{aligned} f''(x) > 0 &\Rightarrow f'(x) \text{ is increasing} \Rightarrow \text{tangent slopes are increasing} \Rightarrow \text{graph of } y = f(x) \text{ bends upward} \\ f''(x) < 0 &\Rightarrow f'(x) \text{ is decreasing} \Rightarrow \text{tangent slopes are decreasing} \Rightarrow \text{graph of } y = f(x) \text{ bends downward} \end{aligned}$$

The manner in which a graph bends or curves is known as **concavity**, and this property is described by the second derivative.

## 2.1 Definitions

**concave up:** if the graph of  $f$  lies above all of its tangents on an interval, then the graph is said to be **concave up** on the interval.

**concave down:** if the graph of  $f$  lies below all of its tangents on an interval, then the graph is said to be **concave down** on the interval.

**inflection point:** an inflection point on the curve  $y = f(x)$  is a point  $(c, f(c))$  at which

- (i)  $f$  is continuous, and
- (ii) the graph of  $y = f(x)$  changes concavity (i.e. changes from concave up to concave down or vice versa.)

So, referring to the graph above, we would say:

- $f$  is concave down on  $(a, p)$  and  $(q, r)$ ;
- $f$  is concave up on  $(p, q)$  and  $(r, b)$ ;
- $f$  has inflection points at  $(p, f(p))$ ,  $(q, f(q))$  and  $(r, f(r))$ .

## 2.2 Concavity Test

Concavity as described by the second derivative is formalized in the **Concavity Test:**

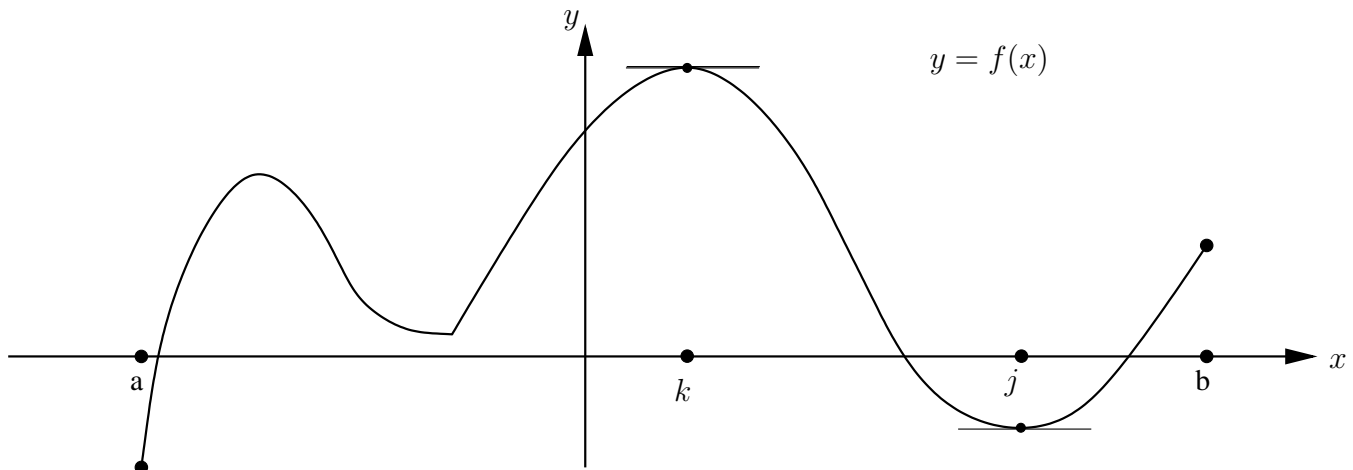
- (i) If  $f''(x) > 0$  on an interval, then the graph of  $y = f(x)$  is concave up on the interval.
- (ii) If  $f''(x) < 0$  on an interval, then the graph of  $y = f(x)$  is concave down on the interval.

Observe on our graph: whenever the graph of  $y = f(x)$  changes from concave up to concave down, or vice versa,  $f'(x)$  changes from increasing to decreasing, or vice versa. That is,  $f''(x)$  changes from positive to negative, or vice versa. This may occur at points where  $f''(x) = 0$  or  $f''(x)$  does not exist.

**Bottom Line:** To determine the intervals of concavity of a function  $f$ :

- (i) Find points at which  $f''$  changes sign (from positive to negative or vice versa).  $f''$  can change sign at points where
  - $f''(x) = 0$
  - $f''(x)$  does not exist
  - $f(x)$  is not defined
- (ii) Test  $f''(x)$  on the intervals defined by the points from (i).

The second derivative can also be used to easily identify when a critical number corresponds to a relative minimum or maximum. Consider the relative maximum at  $x = k$  and the relative minimum at  $x = j$  shown on the following graph and consider  $f'$  and  $f''$  at these two points:



**The Second Derivative Test:** Suppose  $f''(x)$  is continuous near  $x = c$ .

- (i) If  $f'(c) = 0$  and  $f''(c) > 0$  then  $f$  has a relative minimum at  $x = c$ .
- (ii) If  $f'(c) = 0$  and  $f''(c) < 0$  then  $f$  has a relative maximum at  $x = c$ .

### Example 1

Let  $f(x) = x^4 - 2x^3 + 1$ . Determine:

- (i) intervals of increase/decrease
- (ii) local extreme values
- (iii) intervals of concavity
- (iv) inflection points

Sketch the graph of  $y = f(x)$ .

### Example 2

Determine the intervals of concavity and the inflection points of  $f(x) = (x^2 - 4x + 2)e^x$ .