Question 1:

(a)[7 points] Determine the linearization (or linear approximation) to $f(x) = \sqrt[3]{1-4x}$ at $a = 0$.

$$f(a) = f(0) = \sqrt[3]{1-4\cdot(0)} = 1$$

$$f'(x) = \frac{1}{3}(1-4x)^{-2/3}(-4)$$

$$f'(0) = \frac{1}{3}(1-4\cdot\cancel{(0)})^{-2/3}(-4) = \frac{-4}{3}$$

0

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$= 1 - \frac{4}{3}(x-0).$$

$$= \boxed{1 - \frac{4}{3}x}$$

(b)[3 points] Use your result from part (a) to estimate the value of $\sqrt[3]{0.96}$.

$$\sqrt[3]{0.96} = f(0.01)$$

$$\approx L(0.01)$$

$$= 1 - \frac{4}{3}\left(\frac{1}{100}\right)$$

$$= 1 - \frac{\cancel{4}\;1}{\cancel{300}\;75}$$

$$= \boxed{\frac{74}{75}}$$

Question 2:

(a)[3 points] Differentiate $y = \ln\left(x^5 e^x\right)$.

$$y = \ln(x^5) + \ln(e^x)$$
$$= 5\ln x + x$$
$$\boxed{\therefore\ y' = \frac{5}{x} + 1}$$

(b)[3 points] Determine $f'(0)$ if $f(x) = e^{\tan^2 x - \sin x}$.

$$f'(x) = e^{\tan^2 x - \sin x}\left[2\tan x \sec^2 x - \cos x\right]$$
$$f'(0) = \cancel{e^{0^2 - 0}}^{\,1}\left[\cancel{2\cdot 0\cdot 1^2}^{\,0} - 1\right]$$
$$= \boxed{-1}$$

(c)[4 points] Determine $g'(t)$ if $g(t) = 7^{\sqrt{x}} + \log_7\sqrt{1-x}$.

$$g'(t) = 7^{\sqrt{x}}\cdot\ln 7\cdot\tfrac{1}{2}x^{-\frac{1}{2}} + \frac{1}{\sqrt{1-x}\cdot\ln 7}\cdot\tfrac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1).$$
$$= \boxed{\frac{7^{\sqrt{x}}\cdot\ln 7}{2\sqrt{x}} - \frac{1}{2(1-x)\cdot\ln 7}}$$

Question 3:

(a)[5 points] Use logarithmic differentiation to determine y' where $y = \dfrac{x^x}{(x+1)^5}$.

$$y = \frac{x^x}{(x+1)^5}$$

$$\ln y = \ln\left[\frac{x^x}{(x+1)^5}\right]$$

$$\ln y = x\ln x - 5\ln(x+1)$$

$$\therefore \frac{1}{y}y' = \ln x + \cancel{x}\frac{1}{\cancel{x}} - 5\frac{1}{x+1}$$

$$\therefore y' = \frac{x^x}{(x+1)^5}\left[\ln x + 1 - \frac{5}{x+1}\right]$$

(b)[5 points] Determine the equation of the tangent line to curve $1 - xe^y = \ln(x+2y) - y$ at the point $(1, 0)$.

$$\frac{d}{dx}\left[1 - xe^y\right] = \frac{d}{dx}\left[\ln(x+2y) - y\right]$$

$$-e^y - xe^y y' = \frac{1}{x+2y}(1+2y') - y'$$

at $x=1$, $y=0$:

$$-e^0 - 1\cdot e^0 y' = \frac{1}{1+0}(1+2y') - y'$$

$$-1 - y' = 1 + y'$$

$$\therefore 2y' = -2.$$

$$\therefore y' = -1$$

$\therefore$ Equation is $y - 0 = -1(x-1)$ or $\boxed{y = -x+1}$

Question 4: This question deals with the function $f(x) = \ln(1 + x^2) - x$. Note that f has domain all real numbers.

(a) [**5 points**] Determine the intervals of increase and decrease of f and state the relative extrema, if any.

$$f'(x) = \frac{2x}{1+x^2} - 1 = \frac{2x - 1 - x^2}{1+x^2}$$

$$= -\frac{x^2 - 2x + 1}{1+x^2}$$

$$= -\frac{(x-1)^2}{1+x^2}$$

- $f'(x) = 0$? $x = 1$
- $f'(x)$ not exist? no such x.

critical numbers:		1	
test values:	-1		2
$f'(x) = \frac{-(x-1)^2}{1+x^2}$:	$-$	0	$-$
$f(x) = \ln(1+x^2) - x$:	$\searrow$	$\ln 2 - 1$	$\searrow$

$\therefore$ f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

f has no relative extrema.

(b)[**5 points**] Determine the intervals of concavity of the graph of $y = f(x)$ and state the inflection points, if any.

$$f''(x) = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2}$$

$$= \frac{2\left[\,1+x^2 - 2x^2\right]}{(1+x^2)^2}$$

$$= \frac{2\,(1-x)(1+x)}{(1+x^2)^2}$$

- $f''(x) = 0$? $x = 1$, $x = -1$
- $f''(x)$ not exist? no such x.
- $f(x)$ not defined? no such x.

		-1		1	
test values:	-2		0		2
$f''(x) = \dfrac{2(1-x)(1+x)}{(1+x^2)^2}$:	$-$	0	$+$	0	$-$
$f(x) = \ln(1+x^2) - x$:		$\ln 2 + 1$		$\ln 2 - 1$	

$\therefore$ f is concave down on $(-\infty, -1)$ and $(1, \infty)$; concave up on $(-1, 1)$.

f has inflection points at $(-1, \ln 2 + 1)$, $(1, \ln 2 - 1)$.

Question 5 [10 points]: Determine the absolute maximum and minimum values of $f(x) = (x^2-1)^{1/3}$ on the interval $[-3, 1]$.

$$f'(x) = \frac{1}{3}(x^2-1)^{-2/3}(2x) = \frac{2x}{3(x^2-1)^{2/3}}$$

- $f'(x) = 0$? $x = 0$
- $f'(x)$ not exist ? $x = -1, 1$.

Note that f is <u>continuous</u> and $[-3,1]$ is <u>closed</u>.

x	$f(x) = (x^2-1)^{1/3}$
-3	2
-1	0
0	-1
1	0

$\therefore$ f has an absolute maximum of 2 at $x=-3$;

f has an absolute minimum of -1 at $x=0$ and $x=1$.