

Question 1:

(a)[7 points] Determine the linearization (or linear approximation) to $f(x) = \sqrt[3]{1-4x}$ at $a = 0$.

$$f(a) = f(0) = \sqrt[3]{1-4 \cdot (0)} = 1$$

$$f'(x) = \frac{1}{3}(1-4x)^{-\frac{2}{3}}(-4)$$

$$f'(0) = \frac{1}{3}(1-4 \cdot \underset{0}{\cancel{(0)}})^{-\frac{2}{3}}(-4) = -\frac{4}{3}$$

$$\therefore L(x) = f(a) + f'(a)(x-a)$$

$$= 1 - \frac{4}{3}(x-0)$$

$$= \boxed{1 - \frac{4}{3}x}$$

(b)[3 points] Use your result from part (a) to estimate the value of $\sqrt[3]{0.96}$.

$$\sqrt[3]{0.96} = f(0.01)$$

$$\approx L(0.01)$$

$$= 1 - \frac{4}{3}\left(\frac{1}{100}\right)$$

$$= 1 - \frac{4}{300} = 1 - \frac{1}{75}$$

$$= \boxed{\frac{74}{75}}$$

Question 2:

(a)[3 points] Differentiate $y = \ln(x^5 e^x)$.

$$y = \ln(x^5) + \ln(e^x)$$

$$= 5 \ln x + x$$

$$\therefore y' = \frac{5}{x} + 1$$

(b)[3 points] Determine $f'(0)$ if $f(x) = e^{\tan^2 x - \sin x}$.

$$f'(x) = e^{\tan^2 x - \sin x} [2 \tan x \sec^2 x - \cos x]$$

$$f'(0) = e^{0^2 - 0} [2 \cdot 0 \cdot 1^2 - 1]$$

$$= \boxed{-1}$$

(c)[4 points] Determine $g'(t)$ if $g(t) = 7^{\sqrt{x}} + \log_7 \sqrt{1-x}$.

$$g'(t) = 7^{\sqrt{x}} \cdot \ln 7 \cdot \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{\sqrt{1-x} \cdot \ln 7} \cdot \frac{1}{2} (1-x)^{-\frac{1}{2}} \cdot (-1)$$

$$= \boxed{\frac{7^{\sqrt{x}} \cdot \ln 7}{2\sqrt{x}} - \frac{1}{2(1-x) \cdot \ln 7}}$$

Question 3:

(a)[5 points] Use logarithmic differentiation to determine y' where $y = \frac{x^x}{(x+1)^5}$.

$$y = \frac{x^x}{(x+1)^5}$$

$$\ln y = \ln \left[\frac{x^x}{(x+1)^5} \right]$$

$$\ln y = x \ln x - 5 \ln(x+1)$$

$$\therefore \frac{1}{y} y' = \ln x + x \frac{1}{x} - 5 \frac{1}{x+1}$$

$$\therefore y' = \frac{x^x}{(x+1)^5} \left[\ln x + 1 - \frac{5}{x+1} \right]$$

(b)[5 points] Determine the equation of the tangent line to curve $1 - xe^y = \ln(x+2y) - y$ at the point $(1, 0)$.

$$\frac{d}{dx} [1 - xe^y] = \frac{d}{dx} [\ln(x+2y) - y]$$

$$-e^y - xe^y y' = \frac{1}{x+2y} (1+2y') - y'$$

at $x=1, y=0$:

$$-e^0 - 1 \cdot e^0 y' = \frac{1}{1+0} (1+2y') - y'$$

$$-1 - y' = 1 + y'$$

$$\therefore 2y' = -2$$

$$\therefore y' = -1$$

\therefore Equation is $y - 0 = -1(x - 1)$ or

$$y = -x + 1$$

Question 4: This question deals with the function $f(x) = \ln(1+x^2) - x$. Note that f has domain all real numbers.

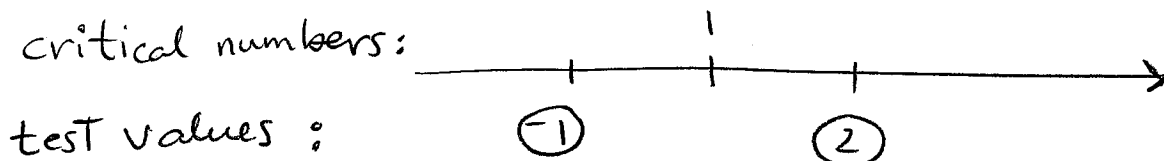
(a) [5 points] Determine the intervals of increase and decrease of f and state the relative extrema, if any.

$$\begin{aligned} f'(x) &= \frac{2x}{1+x^2} - 1 = \frac{2x-1-x^2}{1+x^2} \\ &= -\frac{x^2-2x+1}{1+x^2} \\ &= -\frac{(x-1)^2}{1+x^2} \end{aligned}$$

• $f'(x) = 0?$ $x = 1$

• $f'(x)$ not exist? no such x

critical numbers:



test values :

$$f'(x) = \frac{-(x-1)^2}{1+x^2} :$$

— 0 —

$$f(x) = \ln(1+x^2) - x :$$

↘ $\ln 2 - 1$ ↘

∴ f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

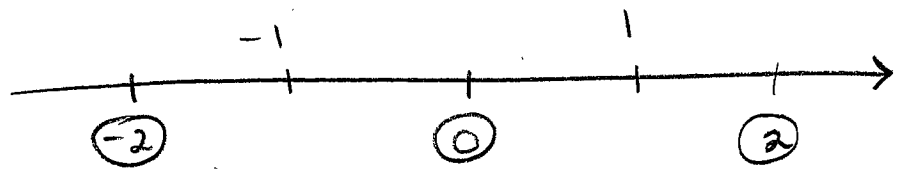
f has no relative extrema.

(b)[5 points] Determine the intervals of concavity of the graph of $y = f(x)$ and state the inflection points, if any.

$$\begin{aligned} f''(x) &= \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} \\ &= \frac{2 [1+x^2 - 2x^2]}{(1+x^2)^2} \\ &= \frac{2(1-x)(1+x)}{(1+x^2)^2} \end{aligned}$$

- $f''(x) = 0$? $x = 1, x = -1$
- $f''(x)$ not exist? no such x .
- $f(x)$ not defined? no such x .

test values:



$$f''(x) = \frac{2(1-x)(1+x)}{(1+x^2)^2} : \quad - \quad 0 \quad + \quad 0 \quad -$$

$$f(x) = \ln(1+x^2) - x : \quad \frown \quad \ln 2 + 1 \quad \smile \quad \ln 2 - 1 \quad \frown$$

$\therefore f$ is concave down on $(-\infty, -1)$ and $(1, \infty)$;
concave up on $(-1, 1)$.

f has inflection points at $(-1, \ln 2 + 1), (1, \ln 2 - 1)$.

Question 5 [10 points]: Determine the absolute maximum and minimum values of $f(x) = (x^2 - 1)^{1/3}$ on the interval $[-3, 1]$.

$$f'(x) = \frac{1}{3} (x^2 - 1)^{-2/3} (2x) = \frac{2x}{3(x^2 - 1)^{2/3}}$$

- $f'(x) = 0$? $x = 0$
- $f'(x)$ not exist ? $x = -1, 1$.

Note that f is continuous and $[-3, 1]$ is closed.

| x | $f(x) = (x^2 - 1)^{1/3}$ |
|-----|--------------------------|
| -3 | 2 |
| -1 | 0 |
| 0 | -1 |
| 1 | 0 |

∴ f has an absolute maximum of 2 at $x = -3$;
 f has an absolute minimum of -1 at $x = 0$
 and $x = 1$.