Question 1 [15 points]: Differentiate the following functions (you do not have to simplify your answers, however points will be deducted for improper use of notation):

(a) [3] $y = 2x^3 + \frac{3}{x} = 2x^3 + 3x^{-1}$

$y' = 6x^2 - 3x^{-2}$

$= \boxed{6x^2 - \frac{3}{x^2}}$

(b) [3] $f(x) = (x^2+x+1)\sqrt{x^2+2} = (x^2+x+1)(x^2+2)^{1/2}$

$f'(x) = (2x+1)(x^2+2)^{\frac{1}{2}} + (x^2+x+1)\frac{1}{2}(x^2+2)^{-\frac{1}{2}}(2x)$

$= \boxed{(2x+1)(x^2+2)^{\frac{1}{2}} + \frac{x(x^2+x+1)}{\sqrt{x^2+2}}}$

(c) [3] $g(t) = \sec\left(\frac{1}{t^2}\right) = \sec(t^{-2})$

$g'(t) = \sec(t^{-2})\tan(t^{-2})(-2t^{-3})$

$= \boxed{\frac{-2\sec(t^{-2})\tan(t^{-2})}{t^3}}$

(d) [3] $y = \frac{4x^3}{\tan x}$

$y' = \frac{(\tan x)(12x^2) - (4x^3)(\sec^2 x)}{(\tan x)^2}$

$= \boxed{\frac{12x^2\tan x - 4x^3\sec^2 x}{\tan^2 x}}$

(e) [3] $f(x) = 5^x x^5$

$\boxed{f'(x) = (5^x \ln 5)x^5 + 5^x(5x^4)}$.

Question 2 [14 points]:

(a) [3] Find $\frac{dy}{dx}$ (you do not have to simplify your answer): $y = (\sin^2 x)\ln(1-x^2)$

$$\frac{dy}{dx} = 2\sin x \cos x \ln(1-x^2) + \sin^2 x \frac{1}{1-x^2}(-2x)$$

$$= \boxed{2\sin x\cos x \ln(1-x^2) - \frac{2x}{1-x^2}\sin^2 x.}$$

(b) [3] Find and simplify $f'(\pi/2)$ where $f(x) = e^{\cos x}$.

$$f'(x) = e^{\cos x}(-\sin x)$$

$$f'\left(\frac{\pi}{2}\right) = e^{\cos\left(\frac{\pi}{2}\right)}\left(-\sin\left(\frac{\pi}{2}\right)\right)$$

$$= e^0 \cdot (-1) = \boxed{-1}$$

(c) [4] Compute $g''(2)$ if $g(t) = \ln(1+\sin(\pi t))$

$$g'(t) = \frac{1}{1+\sin(\pi t)} \cdot \cos(\pi t)\cdot \pi$$

$$= \pi \frac{\cos(\pi t)}{1+\sin(\pi t)}$$

$$g''(t) = \frac{(1+\sin(\pi t))(-\pi\sin(\pi t)\cdot\pi) - \pi\cos(\pi t)\cos(\pi t)\cdot\pi}{(1+\sin(\pi t))^2}$$

$$\therefore g''(2) = \frac{(1+\underbrace{\cancel{\sin(2\pi)}}_{0})(-\pi\underbrace{\cancel{\sin(2\pi)}}_{0}\cdot\pi) - \pi\underbrace{\cancel{\cos(2\pi)}}_{1}\underbrace{\cancel{\cos(2\pi)}}_{1}\cdot\pi}{(1+\underbrace{\cancel{\sin(2\pi)}}_{0})^2} = \boxed{-\pi^2}$$

(d) [4] Find the derivative (you do not have to simplify your answer): $y = \sqrt[3]{e^{2x}x^3}$

$$y = [e^{2x}x^3]^{1/3}$$

$$y' = \boxed{\frac{1}{3}[e^{2x}x^3]^{-\frac{2}{3}} \cdot [2e^{2x}x^3 + e^{2x}\cdot 3x^2]}$$

Question 3 [12 points]: Evaluate the following limits (it may be useful to recall that $\lim_{x\to 0}\frac{\sin x}{x} = 1$):

(a) [3] $\lim_{t\to 1}\frac{t^2-1}{t^2-3t+2}$

$$= \lim_{t\to 1}\frac{\cancel{(t-1)}(t+1)}{\cancel{(t-1)}(t-2)}$$

$$= \frac{2}{-1}$$

$$= \boxed{-2}$$

(b) [3] $\lim_{x\to\infty}\frac{-9x^5-x^2+x}{1+2x^3-3x^5}$

$$= \lim_{x\to\infty}\frac{-9-\frac{1}{x^3}+\frac{1}{x^4}}{\frac{1}{x^5}+\frac{2}{x^2}-3}$$

$$= \frac{-9}{-3}$$

$$= \boxed{3}$$

(c) [3] $\lim_{x\to 1^+} e^{-1/(\sqrt{x}-1)}$

as $x\to 1^+$, $\sqrt{x}-1 \to 0^+$

$\therefore \frac{-1}{\sqrt{x}-1} \to -\infty$.

$$\therefore \lim_{x\to 1^+} e^{\frac{-1}{\sqrt{x}-1}} = \boxed{0}$$

(d) [3] $\lim_{x\to 0}\frac{\tan(3x)}{\sin(5x)}$

$$= \lim_{x\to 0}\frac{\sin(3x)}{\cos(3x)}\cdot\frac{1}{\sin(5x)}$$

$$= \lim_{x\to 0}\frac{\sin(3x)}{3x}\cdot\frac{5x}{\sin(5x)}\cdot\frac{1}{\cos(3x)}\cdot\frac{3\cancel{x}}{5\cancel{x}}$$

$$= 1\cdot 1\cdot\frac{1}{1}\cdot\frac{3}{5}$$

$$= \boxed{\frac{3}{5}}$$

Question 4 [10 points]:

(a) [3] Find the general antiderivative: $f(x) = 4x^3 - 2e^x + \frac{1}{x}$

$$F(x) = \frac{4x^4}{4} - 2e^x + \ln|x| + C$$

$$= \boxed{x^4 - 2e^x + \ln|x| + C}$$

(b) [3] Find the general antiderivative: $f(x) = \frac{2x - \sqrt[3]{x^2} + 7}{x^3}$

$$= 2x^{-2} - x^{-\frac{7}{3}} + 7x^{-3}$$

$$\therefore F(x) = 2\frac{x^{-1}}{(-1)} - \frac{x^{-\frac{4}{3}}}{\left(-\frac{4}{3}\right)} + 7\frac{x^{-2}}{(-2)} + C$$

$$= \boxed{-\frac{2}{x} + \frac{3}{4}\frac{1}{\sqrt[3]{x^4}} - \frac{7}{2}\frac{1}{x^2} + C}$$

(c) [4] A particle has acceleration $a(t) = 6t - 2$ where t is time in seconds. If the initial velocity is $v(0) = 3$ and initial position is $s(0) = 1$, determine the position of the particle at time $t = 2$ seconds.

$$v(t) = \frac{6t^2}{2} - 2t + C$$

$$v(0) = 3 \Rightarrow 0 - 0 + C = 3 \Rightarrow C = 3$$

$$\therefore v(t) = 3t^2 - 2t + 3$$

$$s(t) = \frac{3t^3}{3} - \frac{2t^2}{2} + 3t + C$$

$$s(0) = 1 \Rightarrow 0 - 0 + 0 + C = 1 \Rightarrow C = 1$$

$$\therefore s(t) = t^3 - t^2 + 3t + 1$$

$$\therefore s(2) = 2^3 - 2^2 + 3(2) + 1 = \boxed{11}$$

Question 5 [8 points]: Let $f(x) = \frac{1}{\sqrt{x}}$. Use the <u>definition of the derivative</u> to find $f'(4)$. (No credit will be given if $f'(4)$ is found using differentiation rules.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right]$$

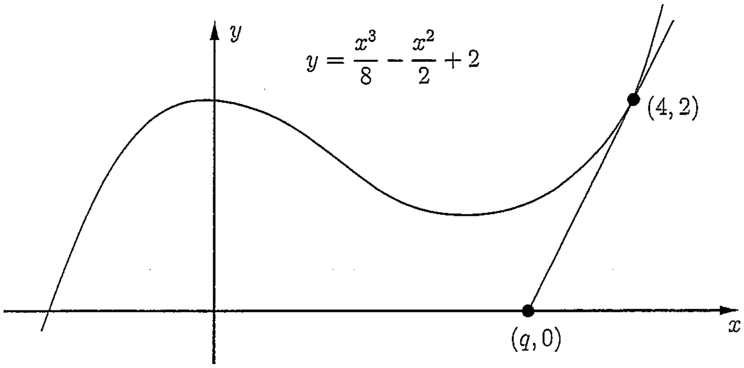
$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{x} - \cancel{x} - \cancel{h}\ {-1}}{\sqrt{x+h}\sqrt{x}\,(\sqrt{x} + \sqrt{x+h})}\right]$$

$$= \frac{-1}{2x\sqrt{x}}$$

$$\therefore\ f'(4) = \frac{-1}{(2)(4)\sqrt{4}} = \boxed{\frac{-1}{16}}$$

Question 6 [10 points]:

(a) [5] Determine the x-intercept q of the tangent line in the following figure.



slope $m = \left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{3x^2}{8} - \frac{2x}{2}\right|_{x=4} = 2.$

also, $m = \frac{2-0}{4-q} = \frac{2}{4-q}$

$\therefore \frac{2}{4-q} = 2$

$4-q = 1$

$\therefore q = 4-1$

$\boxed{q=3}$

(b) [5] Determine the linear approximation of $f(x) = x\sin(\pi x^2)$ at $a = 2$.

$f(a) = f(2) = 2 \cdot \sin(\pi \cdot 2^2) = 0$

$f'(x) = \sin(\pi x^2) + x\cos(\pi x^2)(2\pi x)$

$f'(2) = \underbrace{\sin(\pi \cdot 2^2)}_{0} + 2\underbrace{\cos(\pi \cdot 2^2)}_{1}(2\pi \cdot 2)$

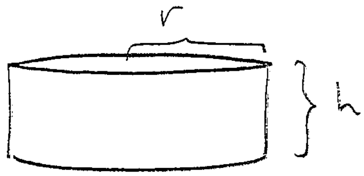
$= 8\pi$

$\therefore L(x) = f(a) + f'(a)(x-a)$

$= 0 + 8\pi(x-2)$

$= \boxed{8\pi x - 16\pi}$

Question 7 [10 points]: A height of a cylinder is decreasing at 2 cm per minute. At what rate is the radius increasing at the instant when the height is 4 cm if the volume is a constant $V = \pi$ cm^3 at all times? State units with your answer. (Recall that the volume of a cylinder is $V = \pi r^2 h$.)



$$\frac{dh}{dt} = -2 \ \text{cm}/\text{min}$$

$$V = \pi \ \text{cm}^3$$

Find $\frac{dr}{dt}$ when $h = 4$ cm.

$$V = \pi r^2 h$$

$$\cancel{\pi} = \cancel{\pi} r^2 h$$

$$r^2 = \frac{1}{h}$$

$$r = h^{-\frac{1}{2}}$$

$$\therefore \frac{dr}{dt} = -\frac{1}{2} h^{-\frac{3}{2}} \frac{dh}{dt}$$

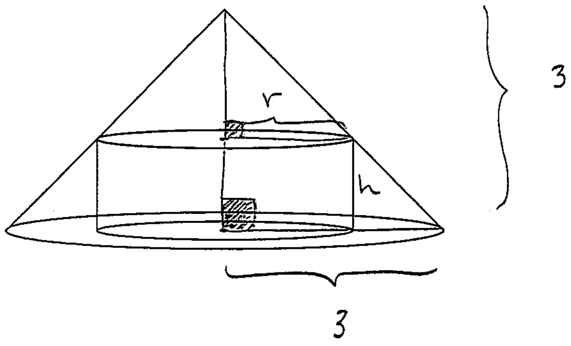
When $h = 4$:

$$\frac{dr}{dt} = +\frac{1}{\cancel{2}} 4^{-\frac{3}{2}} \cdot (+\cancel{2})$$

$$= \frac{1}{8} \ \frac{\text{cm}}{\text{min}}$$

$\therefore$ the radius is increasing at $\frac{1}{8} \frac{\text{cm}}{\text{min}}$.

Question 8 [10 points]: A right circular cylinder is inscribed in a cone of height and base radius both equal to 3 cm. Find the largest possible volume of such a cylinder. Clearly justify all conclusions and state units with your answer. (Recall that the volume of a cylinder is $V = \pi r^2 h$, while that of a cone is $V = \pi r^2 h/3$.)



$V = \pi r^2 h$

By similar triangles, $\frac{3-h}{r} = \frac{3}{3}$

$\therefore r = 3-h$

$\therefore V = \pi (3-h)^2 h = \pi h^3 - 6\pi h^2 + 9\pi h$

Maximize $V(h) = \pi h^3 - 6\pi h^2 + 9\pi h$ on $[0,3]$

$V'(h) = 3\pi h^2 - 12\pi h + 9\pi$

$= 3\pi [h^2 - 4h + 3]$

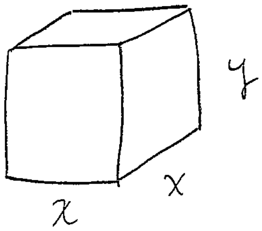
$= 3\pi [h-3][h-1]$

$\therefore h = 3, h = 1$

h	$V(h) = \pi(3-h)^2 h$
0	0
1	$\pi(2)^2 \cdot 1 = 4\pi$
3	0

$\therefore$ The maximum possible volume is 4π cm^3.

Question 9 [10 points]: A box with square base and no top is to have a volume of 6 m^3. Material for the bottom of the box costs \$3 per square metre, while the material for the sides costs \$2 per square metre. Determine the dimensions of the least expensive such box. Clearly justify all conclusions and state units with your answer.



$x^2 y = 6 \text{ m}^3$

Let C = total cost.

$C = 3x^2 + 4 \cdot 2 \cdot xy$

$= 3x^2 + 8xy.$

Minimize $C = 3x^2 + 8xy$ subject to $x^2 y = 6$.

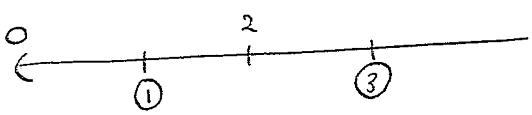
$x^2 y = 6 \Rightarrow y = \frac{6}{x^2}$

$\therefore C(x) = 3x^2 + 8x\left(\frac{6}{x^2}\right) = 3x^2 + \frac{48}{x}$,

So Minimize $C(x) = 3x^2 + \frac{48}{x}$ where $0 < x < \infty$.

$C'(x) = 6x - \frac{48}{x^2} = 6\left(\frac{x^3 - 8}{x^2}\right)$

$C'(x) = 0 \Rightarrow x = 2.$



$C'(x) = \frac{6(x^3-8)}{x^2}$: $-$ 0 $+$

$C(x)$: $\searrow$ $\nearrow$

$\therefore$ Cost $C(x)$ has a local and absolute minimum at $x = 2$.

$\therefore$ Dimensions of the least expensive box are $x = 2$ & $y = \frac{6}{x^2} = \frac{6}{4} = \frac{3}{2}$ m.

Question 10 [10 points]:

(a) [5] Determine the equation of the tangent line to the curve

$$\sqrt{x+y} = 3 + x^2y^2$$

at the point $(0,9)$. Implicit differentiation may help here.

$$\frac{d}{dx}\left[(x+y)^{\frac{1}{2}}\right] = \frac{d}{dx}\left[3 + x^2y^2\right]$$

$$\frac{1}{2}(x+y)^{-\frac{1}{2}}(1+y') = 2xy^2 + x^2 2yy'$$

at $x=0, y=9$: $\frac{1}{2}(0+9)^{-\frac{1}{2}}(1+y') = 0 + 0$

$\therefore \quad 1+y' = 0$

$y' = -1$

$\therefore$ Equation of tangent line is

$y - 9 = (-1)(x-0)$

or $\boxed{y = -x + 9}$

(b) [5] Use logarithmic differentiation to find $\frac{dy}{dx}$:

$$y = \sqrt{x}e^{x^2}(x^2+1)^{10} = x^{\frac{1}{2}} e^{x^2} (x^2+1)^{10}$$

$$\ln y = \frac{1}{2}\ln x + x^2 + 10\ln(x^2+1)$$

$$\frac{1}{y}y' = \frac{1}{2}\frac{1}{x} + 2x + \frac{10}{x^2+1}(2x)$$

$$\therefore \quad y' = \sqrt{x}e^{x^2}(x^2+1)^{10}\left[\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right].$$

Question 11 [16 points]: For this question consider the function $f(x) = \dfrac{x^2+12}{x-2}$.

(a) [1] Determine the domain of $f(x)$.

Domain is $(-\infty, 2) \cup (2, \infty)$.

(b) [2] Determine the x and y intercepts of the graph of $y = f(x)$.

$f(x) = 0 \Rightarrow x^2+12=0$ } no solutions, so no x-intercept.

y-intercept is at $(0, f(0)) = \left(0, \frac{12}{-2}\right) = (0, -6)$.

(c) [4] Determine the intervals of increase and decrease of $f(x)$.

$$f'(x) = \frac{(x-2)(2x) - (x^2+12)(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2 - 12}{(x-2)^2}$$

$$= \frac{x^2-4x-12}{(x-2)^2}$$

$$= \frac{(x-6)(x+2)}{(x-2)^2}$$

$f'(x) = 0$ at $x = 6, -2$

$f(x)$ is not defined at $x = 2$

	-3	-2	0	2	3	6	8
$f'(x) = \frac{(x-6)(x+2)}{(x-2)^2}$:	$+$	0	$-$	NA	$-$	0	$+$
$f(x) = \frac{x^2+12}{x-2}$:	$\nearrow$	-4	$\searrow$	NA	$\searrow$	12	$\nearrow$

$\therefore$ $f(x)$ is increasing on $(-\infty, -2) \cup (6, \infty)$.

$f(x)$ is decreasing on $(-2, 2) \cup (2, 6)$.

(d) [1] State the local maxima and minima of $f(x)$.

f has a loc. max. of -4 at $x = -2$.

f has a loc. min. of 12 at $x = 6$.

continued on next page...

(e) [2] Using the fact that $f''(x) = \dfrac{32}{(x-2)^3}$ determine the intervals of concavity of $f(x)$ and identify inflection points, if any.

	0	2	3
$f''(x) = \dfrac{32}{(x-2)^3}$:	$-$	NA	$+$
$f(x)$:	$\frown$	NA	$\smile$

$\therefore$ f is concave up on $(2, \infty)$.
f is concave down on $(-\infty, 2)$.

(f) [2] Determine the horizontal and vertical asymptotes, if any.

(i) $\lim\limits_{x \to \infty} \dfrac{x^2+12}{x-2} = \infty$, $\lim\limits_{x \to -\infty} \dfrac{x^2+12}{x-2} = -\infty$,

so no horizontal asymptotes.

(ii) • $\lim\limits_{x \to 2^+} \dfrac{x^2+12}{x-2} = +\infty$, so $x = 2$ is a vert. asymptote.

• $\lim\limits_{x \to 2^-} \dfrac{x^2+12}{x-2} = -\infty$, so again $x = 2$ is a vert. asymptote.

(g) [4] Use the information gathered in parts (a)-(f) to make an informative sketch of the graph of $y = f(x)$. Label your axes and any of the interesting points on your graph (intercepts, local extrema, etc.)