

Question 1:

(a) [7 points] Determine the linearization (or linear approximation) to $f(x) = \frac{1}{\sqrt{1+3x}}$ at $a = 0$.

$$f(a) = f(0) = \frac{1}{\sqrt{1+3 \cdot 0}} = 1$$

$$f'(x) = -\frac{1}{2} (1+3x)^{-\frac{3}{2}} \quad (3)$$

$$f'(a) = f'(0) = -\frac{3}{2}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$= 1 - \frac{3}{2}(x-0)$$

$$= 1 - \frac{3}{2}x$$

$$\therefore L(x) = 1 - \frac{3}{2}x$$

(b) [3 points] Use your result from part (a) to estimate the value of $\frac{1}{\sqrt{1.03}}$.

$$\begin{aligned} \frac{1}{\sqrt{1.03}} &= f(0.01) \approx L(0.01) \\ &= 1 - \frac{3}{2} \left(\frac{1}{100} \right) \\ &= 1 - \frac{3}{200} \\ &= \boxed{\frac{197}{200}} \end{aligned}$$

Question 2:

(a) [3 points] Differentiate $y = e^{x^5 - \ln(x^2)}$.

$$y' = e^{x^5 - \ln(x^2)} \left[5x^4 - \frac{1}{x^2} (2x) \right]$$

$$= e^{x^5 - \ln(x^2)} \left[5x^4 - \frac{2}{x} \right]$$

(b) [3 points] Determine $f'(0)$ if $f(x) = \ln\left(\frac{e^x}{x^2+1}\right)$.

$$f(x) = \ln\left(\frac{e^x}{x^2+1}\right) = \ln(e^x) - \ln(x^2+1)$$

$$= x - \ln(x^2+1)$$

$$\therefore f'(x) = 1 - \frac{1}{1+x^2} (2x)$$

$$f'(0) = 1 - \frac{1}{1+0^2} (2 \cdot 0)$$

$$= \boxed{1}$$

(c) [4 points] Determine $g'(x)$ if $g(x) = 5^{\sqrt{1-x}} + \log_5 \sqrt{x}$.

$$g'(x) = 5^{\sqrt{1-x}} \ln 5 \cdot \frac{1}{2} (1-x)^{-\frac{1}{2}} (-1) + \frac{1}{\sqrt{x} \ln 5} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{-5^{\sqrt{1-x}} \ln 5}{2\sqrt{1-x}} + \frac{1}{2x \cdot \ln 5}$$

Question 3:

(a) [5 points] Use logarithmic differentiation to determine y' where $y = \frac{(x-1)^7}{x^{2x}}$.

$$y = \frac{(x-1)^7}{x^{2x}}$$

$$\ln y = 7 \ln(x-1) - 2x \ln x$$

$$\frac{1}{y} y' = \frac{7}{x-1} - 2 \ln x - 2x \cdot \frac{1}{x}$$

$$\therefore y' = \frac{(x-1)^7}{x^{2x}} \left[\frac{7}{x-1} - 2 \ln x - 2 \right]$$

(b) [5 points] Determine the equation of the tangent line to curve $y + x^2 e^y = 1 + \ln(x + 3y)$ at the point $(1, 0)$.

$$\frac{d}{dx} [y + x^2 e^y] = \frac{d}{dx} [1 + \ln(x + 3y)]$$

$$y' + 2x e^y + x^2 e^y y' = \frac{1}{x+3y} (1+3y')$$

at $x=1, y=0$:

$$y' + 2(1)e^0 + 1^2 e^0 y' = \frac{1}{1+3 \cdot 0} (1+3y')$$

$$2y' + 2 = 1 + 3y'$$

$$y' = 1$$

$$\therefore y - 0 = 1(x - 1)$$

$$\text{or } \boxed{y = x - 1}$$

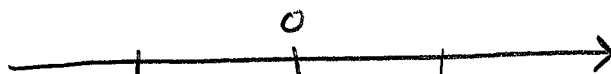
Question 4: This question deals with the function $f(x) = e^{-\frac{1}{2}x^2}$. Note that f has domain all real numbers.

(a)[5 points] Determine the intervals of increase and decrease of f and state the relative extrema, if any.

$$\begin{aligned} f'(x) &= e^{-\frac{1}{2}x^2} \left(-\frac{1}{2} \cdot 2x\right) \\ &= -e^{-\frac{1}{2}x^2} x \end{aligned}$$

- $f'(x) = 0$? $x = 0$
- $f'(x)$ not exist ? no such x .

crit. num:



test values:

⊖

⊕

$$f'(x) = -e^{-\frac{1}{2}x^2} x : \quad + \quad 0 \quad -$$

$$f(x) = e^{-\frac{1}{2}x^2} : \quad \nearrow \quad 1 \quad \searrow$$

∴ f is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$.

f has a rel. max. of 1 at $x=0$.

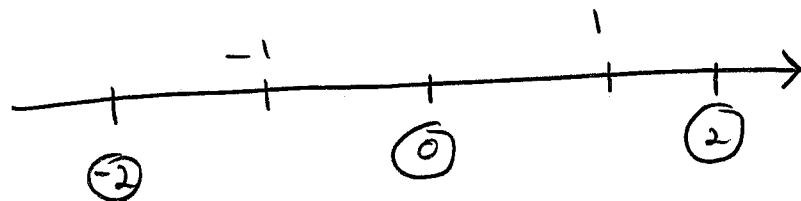
(b)[5 points] Determine the intervals of concavity of the graph of $y = f(x)$ and state the inflection points, if any.

$$f'(x) = -xe^{-\frac{1}{2}x^2}$$

$$\begin{aligned} f''(x) &= -e^{-\frac{1}{2}x^2} - xe^{-\frac{1}{2}x^2}(-x) \\ &= -e^{-\frac{1}{2}x^2} [1 - x^2] \end{aligned}$$

$$\begin{aligned} \bullet f''(x) = 0? \quad & 1 - x^2 = 0 \\ & (1 - x)(1 + x) = 0 \\ & x = 1, \quad x = -1 \end{aligned}$$

$$\left. \begin{array}{l} \bullet f''(x) \text{ not exist?} \\ \bullet f(x) \text{ not defined?} \end{array} \right\} \text{no such } x.$$



test values:

$$f''(x) = -e^{-\frac{1}{2}x^2} (1 - x^2): \quad + \quad 0 \quad - \quad +$$

$$f(x) = e^{-\frac{1}{2}x^2} \quad : \quad \cup \quad e^{-\frac{1}{2}} \quad \cap \quad e^{-\frac{1}{2}} \quad \cup$$

\therefore graph of $y = f(x)$ is concave up on $(-\infty, -1)$, $(1, \infty)$;
concave down on $(-1, 1)$.

Graph has inflection points at $(-1, e^{-\frac{1}{2}})$, $(1, e^{-\frac{1}{2}})$.

Question 5 [10 points]: Determine the absolute maximum and minimum values of $f(x) = (x^2 + 2x)^{1/3}$ on the interval $[-2, 2]$.

f is continuous and $[-2, 2]$ is closed.

$$\begin{aligned} f'(x) &= \frac{1}{3} (x^2 + 2x)^{-2/3} (2x + 2) \\ &= \frac{2(x+1)}{3 [x(x+2)]^{2/3}} \end{aligned}$$

- $f'(x) = 0$ at $x = -1$
- $f'(x)$ does not exist at $x = 0$, $x = -2$.

x	$f(x) = (x^2 + 2x)^{1/3}$
-2	0
-1	-1
0	0
2	2

- ∴ f has an abs. max. of 2 at $x = 2$;
 f has an abs. min. of -1 at $x = -1$.