Question 1: [**45 points**] This question consists of 15 short answer problems each worth 3%. For each problem, clearly write your final answer in the box to the right. The solution to each problem is short, requiring no more space than that given.

(a) Find $f^{-1}(x)$ if $f(x) = \sqrt[3]{1-3x} + 2$.

$y = (1-3x)^{1/3} + 2$

$x = (1-3y)^{1/3} + 2$

$x - 2 = (1-3y)^{1/3}$

$(x-2)^3 = 1 - 3y$

$3y = 1 - (x-2)^3$

$y = \dfrac{1-(x-2)^3}{3}$

$\boxed{f^{-1}(x) = \dfrac{1-(x-2)^3}{3}}$

(b) Solve for x: $4^{2x-1} - 3 = 61$.

$4^{2x-1} = 64$

$4^{2x-1} = 4^3$

$\therefore\ 2x - 1 = 3$

$x = 2$

$\boxed{x = 2}$

(c) A population grows according to the model $N(t) = 3000e^{0.12t}$, where $N(t)$ is the population at time t years and $t = 0$ corresponds to the present. Find the doubling time of the population. (Round your answer to 1 decimal place.)

$3000e^{0.12t} = 6000$

$e^{0.12t} = \dfrac{6000}{3000} = 2$

$\therefore\ 0.12t = \ln(2)$

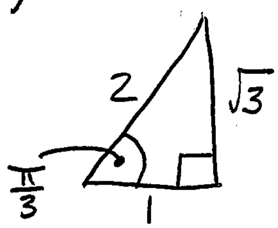
$t = \dfrac{\ln(2)}{0.12} \doteq 5.8$

$\boxed{t \doteq 5.8 \text{ yrs.}}$

(d) Compute $\tan(22\pi/3)$ exactly.

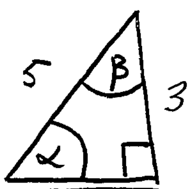
$\boxed{\sqrt{3}}$

$$\tan\left(\frac{22\pi}{3}\right)$$
$$= \tan\left(\frac{21\pi}{3} + \frac{\pi}{3}\right)$$
$$= \tan\left(7\pi + \frac{\pi}{3}\right)$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{1}$$



(e) A right triangle has acute angles α and β. If $\sec\beta = 5/3$, what is $\cot\alpha$?

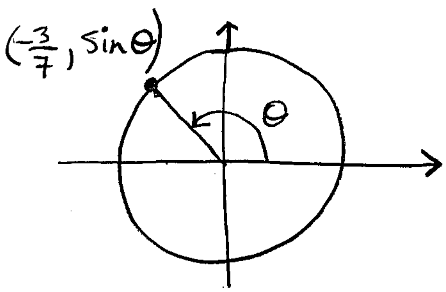
$\boxed{\cot(\alpha) = \frac{4}{3}}$



$$\therefore \sqrt{5^2 - 3^2} = 4$$
$$\therefore \cot(\alpha) = \frac{4}{3}$$

(f) If $\cos\theta = -3/7$ where $0 \le \theta \le \pi$, find the exact value of $\cos\left(\theta + \frac{\pi}{3}\right)$.

$\boxed{\frac{-3 - 2\sqrt{30}}{14}}$



$$\therefore \sin(\theta) \pm \sqrt{1 - \cos^2\theta}$$
$$= \sqrt{1 - \left(\frac{-3}{7}\right)^2}$$
$$= \sqrt{\frac{49 - 9}{49}} = \frac{2\sqrt{10}}{7}$$

$$\therefore \cos\left(\theta + \frac{\pi}{3}\right)$$
$$= \cos(\theta)\cos\left(\frac{\pi}{3}\right) - \sin(\theta)\sin\left(\frac{\pi}{3}\right)$$
$$= \left(\frac{-3}{7}\right)\left(\frac{1}{2}\right) - \left(\frac{2\sqrt{10}}{7}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-3 - 2\sqrt{30}}{14}$$

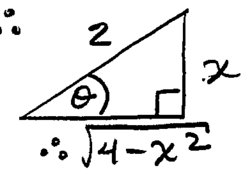
(g) Express $\sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)$ in terms of x (without trigonometric functions).

$$\sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right)$$
$$= 2\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$$
$$= 2\left(\frac{x}{2}\right)\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$$
$$= 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) = \frac{x\sqrt{4-x^2}}{2}$$

$$\boxed{\frac{x\sqrt{4-x^2}}{2}}$$

let $\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\therefore$

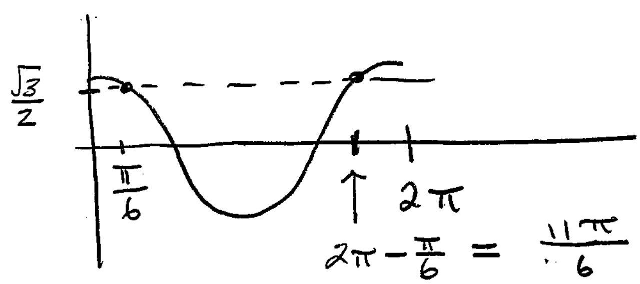


$\therefore \cos(\theta) = \frac{\sqrt{4-x^2}}{2}$

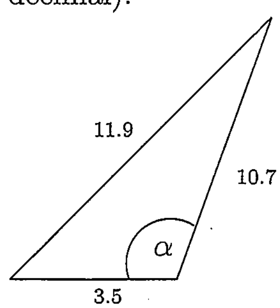
(h) Find all solutions in $[0, 2\pi]$ to $2\cos x - \sqrt{3} = 0$.

$$2\cos x - \sqrt{3} = 0$$
$$\therefore \cos x = \frac{\sqrt{3}}{2}$$
$$\therefore x = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$\boxed{x = \frac{\pi}{6}, \frac{11\pi}{6}}$$



(i) Find the value of α in the following triangle (give your answer in degrees, rounded to one decimal):



$$\boxed{\alpha \doteq 101.5^\circ}$$

$$(11.9)^2 = (10.7)^2 + (3.5)^2 - 2(10.7)(3.5)\cos(\alpha)$$
$$\therefore \alpha = \cos^{-1}\left[\frac{(11.9)^2 - (10.7)^2 - (3.5)^2}{-2(10.7)(3.5)}\right]$$
$$\doteq 101.5$$

(j) Reduce the augmented coefficient matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 1 \\ 1 & -2 & 4 & -5 \end{array}\right]$ to determine how many solutions the system has. (The answer is either zero, exactly one, or infinitely many solutions; which is it)?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 1 \\ 1 & -2 & 4 & -5 \end{array}\right]$$

new $R_3 = (-1)R_1 + R_3$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 4 & -2 \end{array}\right]$$

new $R_3 = 2R_2 + R_3$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ infinitely many solutions

$\boxed{\text{infinitely many.}}$

(k) Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -5 \\ -9 & 2 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} -2 & 2 \\ 4 & -1 \end{bmatrix}$. Compute $\mathbf{A}(\mathbf{B}+2\mathbf{C})$.

$$B + 2C = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$$

$\therefore A(B+2C)$

$$= \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

$\boxed{\begin{bmatrix} -1 & -2 \end{bmatrix}}$

(l) Suppose $\mathbf{A}$ is size 4×1, $\mathbf{B}$ is size 6×4, $\mathbf{C}$ is size 4×4, and $\mathbf{D}$ is size 1×6. What is the size of the product $\mathbf{CADB}$?

C A D B

$(4\times4)(4\times1)(1\times6)(6\times4)$

$\therefore 4 \times 4$

$\boxed{4 \times 4}$

(m) An arithmetic sequence has $a_{14} = 19$ and $a_{24} = -11$. What is a_2?

$$d = \frac{a_{24} - a_{19}}{24-14} = \frac{-11-19}{10} = -3$$

$$\boxed{a_2 = 55}$$

$$a_{14} = a_1 + (14-1)d$$

$$\therefore\ 19 = a_1 + (13)(-3)$$

$$\therefore\ a_1 = 19 + 39 = 58$$

$$\therefore\ a_2 = a_1 + d = 58 + (-3) = 55$$

(n) Compute the sum of the arithmetic series $\sum_{k=1}^{29}\left(5 - \frac{k}{2}\right)$.

$$a_n = 5 - \frac{n}{2}$$

$$\boxed{-\frac{145}{2}}$$

$$a_1 = 5 - \frac{1}{2} = \frac{9}{2}$$

$$a_{29} = 5 - \frac{29}{2} = \frac{-19}{2}$$

$$\therefore\ S_{29} = \frac{29}{2}\left[\frac{9}{2} + \frac{-19}{2}\right] = \left(\frac{29}{2}\right)(-5) = \frac{-145}{2}$$

(o) Find the 15th term of the geometric sequence $-\frac{5}{81}, \frac{5}{54}, \ldots$.

$$a_1 = \frac{-5}{81}$$

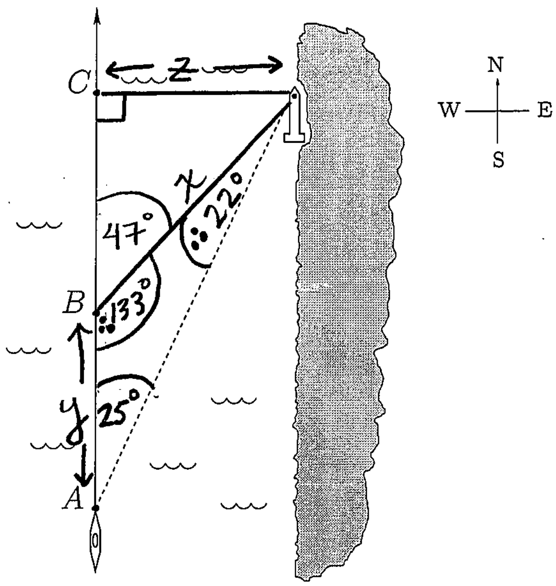
$$\boxed{\frac{-5 \cdot 3^{10}}{2^{14}} = \frac{-295,245}{16,384}}$$

$$r = \frac{\left(\frac{5}{54}\right)}{\left(\frac{-5}{81}\right)} = -\frac{3}{2}$$

$$\therefore\ a_{15} = a_1 r^{15-1} = \left(\frac{-5}{81}\right)\left(\frac{-3}{2}\right)^{14} = \frac{-5}{3^4} \cdot \frac{3^{14}}{2^{14}} = \frac{-5 \cdot 3^{10}}{2^{14}}$$

Question 2

A kayaker is traveling due north at 4 km/h along a rocky coastline. At point A the kayaker spots a lighthouse on the shore at a compass bearing of 25°. The boater takes a second compass bearing 1.5 hours later at B and finds the lighthouse to be at 47°. (Note: figure not to scale.)



(a)[7 points]: What is the distance between the kayaker and the lighthouse when the second reading is taken? (Round your answer to 2 decimal places and give units.)

$$y = \left(4 \frac{km}{h}\right)\left(1.5\ h\right) = 6\ km$$

$$\therefore \frac{\sin(22^\circ)}{y} = \frac{\sin(25^\circ)}{x}$$

$$\therefore x = \frac{y \sin(25^\circ)}{\sin(22^\circ)} = \frac{(6)\sin(25^\circ)}{\sin(22^\circ)} \doteq \boxed{6.77\ km}$$

(b)[3 points]: As the kayaker continues north beyond point B, he eventually reaches a point C at which his distance from the lighthouse is a minimum. Find this minimum distance. (Again, round your answer to 2 decimal places and give units.)

$$\sin(47^\circ) = \frac{z}{x}$$

$$\therefore z = x\sin(47^\circ) = \frac{(6)\sin(25^\circ)\sin(47^\circ)}{\sin(22^\circ)}$$

$$\doteq \boxed{4.95\ km}$$

Question 3 [7 points]:
Solve for x:

$$\ln(x) = \ln(2x-1) - \ln(x-2)$$

$$\ln(x) = \ln\left(\frac{2x-1}{x-2}\right)$$

$$\therefore \quad x = \frac{2x-1}{x-2}$$

$$x^2 - 2x = 2x - 1$$

$$x^2 - 4x + 1 = 0$$

$$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

Check: $x = 2 + \sqrt{3}$:

$$\ln(2+\sqrt{3}) \quad \ln(2(2+\sqrt{3})-1) - \ln(2+\sqrt{3}-2)$$

$$\ln(2+\sqrt{3}) \quad \ln(4+2\sqrt{3}-1) - \ln(\sqrt{3})$$

$$\ln(2+\sqrt{3}) \quad \ln\left(\frac{3+2\sqrt{3}}{\sqrt{3}}\right)$$

$$\ln(2+\sqrt{3}) = \ln(\sqrt{3}+2) \quad \checkmark$$

$x = 2 - \sqrt{3}$: $\ln(2-\sqrt{3}) \quad \ln(2(2-\sqrt{3})-1) - \ln(2-\sqrt{3}-2)$

$$\ln(4-2\sqrt{3}-1) - \underbrace{\ln(-\sqrt{3})}_{\text{not defined !}}$$

$\therefore x = 2 + \sqrt{3}$ is the only solution

Question 4

(a)[7 points]: Carefully (and neatly!) sketch the graph of

$$y = 2\cos\left(2x - \frac{\pi}{2}\right) - 1 = 2\cos\left[2\left(x - \frac{\pi}{4}\right)\right] - 1$$

Clearly indicate the scale on the x and y axes.

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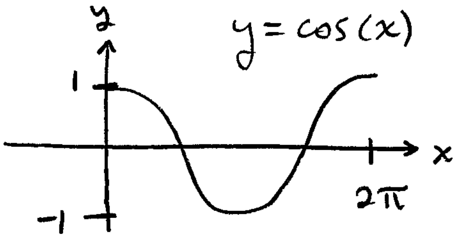

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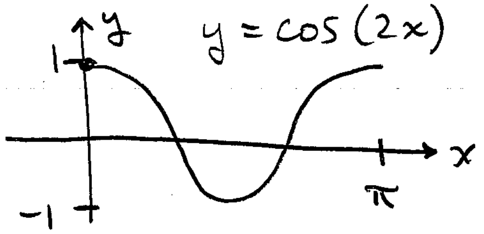

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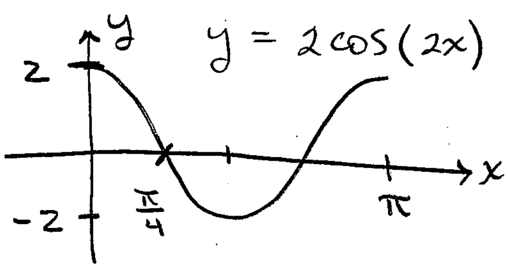

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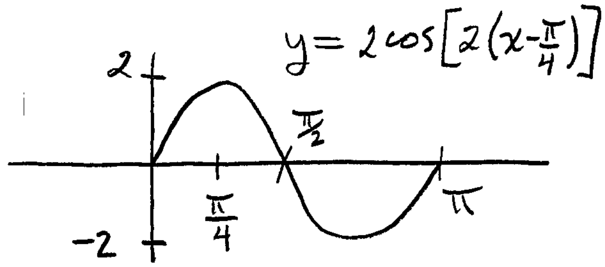

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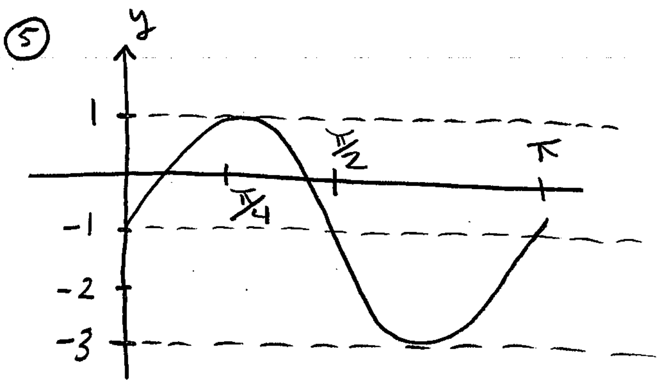


$y = 2\cos\left[2\left(x - \frac{\pi}{4}\right)\right] - 1$

(b)[3 points]: State the period, phase-shift and amplitude of the function in (a).

Period $= \frac{2\pi}{2} = \pi$

Phase Shift $= \frac{\pi}{4}$

Amplitude $= 2$

Question 5

(a)[7 points]: Find $\mathbf{A}^{-1}$, where $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$.

① $\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$

② new $R_3 = (-2)R_1 + R_3$:

$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right]$

③ new $R_2 = (\frac{1}{2})R_2$

$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right]$

④ new $R_1 = R_2 + R_1$:
new $R_3 = (-5)R_2 + R_3$:

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -2 & -\frac{5}{2} & 1 \end{array}\right]$

⑤ new $R_3 = 2R_3$:

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right]$

⑥ new $R_1 = (-\frac{1}{2})R_3 + R_1$
new $R_2 = (\frac{1}{2})R_3 + R_2$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right]$

$\boxed{\therefore A^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}}$

(b)[3 points]: Use your result in (a) to solve the system of equations

$$\left.\begin{aligned} x - y + z &= -2 \\ 2y - z &= 4 \\ 2x + 3y &= 3 \end{aligned}\right\} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -6 \end{bmatrix}$; $\therefore x = 3$
$y = -1$
$z = -6$.

Question 6 [8 points]:
Find all values of θ in $[0, 2\pi]$ for which

$$4\sin^2\theta = 3 - 4\sin\theta\ .$$

$$4\sin^2\theta + 4\sin\theta - 3 = 0$$

let $u = \sin\theta$: $4u^2 + 4u - 3 = 0$

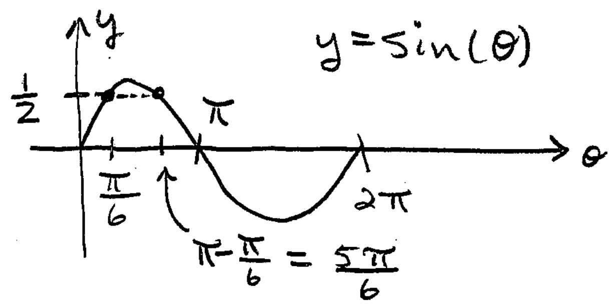
$$\therefore\ u = \frac{-4 \pm \sqrt{16 - 4(4)(-3)}}{2(4)}$$

$$= \frac{-4 \pm 8}{8}$$

$$= \frac{1}{2}, \ -\frac{3}{2}$$

$\therefore$ $\underbrace{\sin(\theta) = -\frac{3}{2}}$; $\underbrace{\sin(\theta) = \frac{1}{2}}$

no solution since
$-1 \le \sin(\theta) \le 1$
for all real θ.



$\therefore$ $\sin(\theta) = \frac{1}{2}$ at

$$\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}.$$

Question 7

After graduation a student goes to work for a mining company in a remote location. Because of the isolation there is no opportunity for recreation, so the employees work every day of the year, followed by six months of vacation. The mining company has difficulty attracting employees under these conditions, so they allow new employees to negotiate their salary. Having paid close attention in math class, the clever student presents the following unconventional salary proposal: the student requests to be paid \$2 for the first day of work, $\$2(1.025) = \$2.05$ for the second day, $\$2(1.025)^2 = \$2.10125$ for the third day, and so on, increasing the pay by 2.5% each day for all 365 days of the year. (For simplicity, we assume that the exact amount earned each day can be paid; the pay is not rounded off to the nearest cent.) The employer accepts the proposal.

(a) [**2 points**]: Find a formula for a_n, the amount of pay received for working the n^{th} day of the year.

$$a_n = 2(1.025)^{n-1}$$

(b) [**2 points**]: How much pay was received for working the last day of the year? (Round your answer to the nearest dollar.)

$$a_{365} = 2(1.025)^{365-1} \doteq \$16,015$$

(c) [**3 points**]: What is the total amount of pay received for the year? (Round your answer to the nearest dollar.)

$$S_{365} = \sum_{k=1}^{365} 2(1.025)^{k-1} \quad \left.\begin{matrix}\text{geometric series}, r = 1.025, \\ a_1 = 2, \quad n = 365\end{matrix}\right\}$$

$$\therefore S_{365} = \frac{2(1-1.025^{365})}{1-1.025} \doteq \$656,520$$

(d) [**3 points**]: How many days does it take the new worker to earn the first \$100,000?

Solve $S_n = 100,000$ for n:

$$\therefore \frac{2(1-1.025^n)}{1-1.025} = 100,000$$

$$\therefore 1 - 1.025^n = \frac{100,000(1-1.025)}{2}$$

$$\therefore 1.025^n = 1 - \frac{100000(1-1.025)}{2}$$

$$\therefore n = \frac{\ln\left[1 - \frac{100,000(1-1.025)}{2}\right]}{\ln(1.025)}$$

$$\doteq \boxed{289 \text{ days.}}$$