Math 121 Sec S09N01 - Test 1
Jan 30 2009

Question 1:

(a)[7 points] Let $f(x) = \dfrac{x^2}{1+x^2}$ and $\sqrt{x-2}$. Compute and simplify $(f \circ g)(x)$ and state the domain.

$$\left.(f \circ g)(x) = f(g(x)) = \frac{(\sqrt{x-2})^2}{1+(\sqrt{x-2})^2}\right\} \ *$$

$$= \frac{x-2}{1+x-2}$$

$$= \frac{x-2}{x-1}$$

Using $*$, must have $x-2 \geq 0$, i.e. $x \geq 2$

$\therefore$ domain of $f \circ g$ is $[2, \infty)$.

(b)[3 points] Let $H(x) = \dfrac{\sin^2(x)}{\sqrt{1-\sin^2(x)}}$. If $g(x) = \sin(x)$ and $H = f \circ g$, what is $f(x)$?

$$H(x) = \frac{(g(x))^2}{\sqrt{1-(g(x))^2}}$$

$$\therefore \ f(x) = \frac{x^2}{\sqrt{1-x^2}}$$

Question 2:

(a)[**5 points**] Evaluate the following limit, if it exists: $\lim_{x\to -2} \frac{x^3 - 2x + 8}{x^2 - 2}$

$$\lim_{x\to -2} \frac{x^3-2x+8}{x^2-2}$$

$$= \frac{(-2)^3 - 2(-2) + 8}{(-2)^2 - 2}$$

$$= \frac{-8+4+8}{4-2}$$

$$= 2$$

(b)[**5 points**] Evaluate the following limit, if it exists: $\lim_{t\to 5} \frac{t^2 - t - 20}{t^2 - 9t + 20}$

$$\lim_{t\to 5} \frac{t^2-t-20}{t^2-9t+20}$$

$$= \lim_{t\to 5} \frac{\cancel{(t-5)}(t+4)}{\cancel{(t-5)}(t-4)}$$

$$= \frac{9}{1}$$

$$= 9$$

Question 3:

(a) [5 points] Evaluate the following limit, if it exists: $\lim_{x\to 4} \frac{1-\sqrt{5-x}}{4-x}$

$$\lim_{x\to 4} \frac{1-\sqrt{5-x}}{4-x}$$

$$= \lim_{x\to 4} \frac{1-\sqrt{5-x}}{4-x} \cdot \frac{1+\sqrt{5-x}}{1+\sqrt{5-x}}$$

$$= \lim_{x\to 4} \frac{1-(5-x)}{(4-x)\left(1+\sqrt{5-x}\right)}$$

$$= \lim_{x\to 4} \frac{-\cancel{(4-x)}}{\cancel{(4-x)}\left(1+\sqrt{5-x}\right)}$$

$$= \frac{-1}{2}$$

(b) [5 points] Evaluate the following limit, if it exists: $\lim_{t\to \pi^-} \frac{\cos(t)}{t-\pi}$

As $t \to \pi^-$, $\cos(t) \to -1^+$ and $t - \pi \to 0^-$

$$\therefore \lim_{t\to\pi^-} \frac{\cos(t)}{t-\pi} = \infty$$

Question 4:

(a) [5 points] Evaluate the following limit, if it exists: $\lim\limits_{x\to\infty} \dfrac{5x^4 - 7x^2 + \pi}{-11x^4 + 7x^3 - 1}$

$$\lim_{x\to\infty} \frac{5x^4 - 7x^2 + \pi}{-11x^4 + 7x^3 - 1} \begin{matrix} \div x^4 \\ \div x^4 \end{matrix}$$

$$= \lim_{x\to\infty} \frac{5 - \frac{7}{x^2} + \frac{\pi}{x^4}}{-11 + \frac{7}{x} - \frac{1}{x^4}}$$

$$= -\frac{5}{11}$$

(b) [5 points] Evaluate the following limit, if it exists: $\lim\limits_{\theta\to 0} \dfrac{\theta + \sin(3\theta)}{\theta - \sin(3\theta)}$

$$\lim_{\theta\to 0} \frac{\theta + \sin(3\theta)}{\theta - \sin(3\theta)} \begin{matrix} \div 3\theta \\ \div 3\theta \end{matrix}$$

$$= \lim_{\theta\to 0} \frac{\frac{\cancel{\theta}}{3\cancel{\theta}} + \frac{\sin(3\theta)}{(3\theta)}}{\frac{\cancel{\theta}}{3\cancel{\theta}} - \frac{\sin(3\theta)}{(3\theta)}}$$

$$= \frac{\frac{1}{3} + 1}{\frac{1}{3} - 1}$$

$$= \frac{\left(\frac{4}{3}\right)}{\left(-\frac{2}{3}\right)} = -2$$

Question 5:

(a) [**6 points**] Use the definition of the derivative to find $f'(x)$ if $f(x) = \frac{1}{x^2}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{(x+h)^2 x^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{(x+h)^2 x^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(-2x-h)}{(x+h)^2 x^2}\right]$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}$$

(b) [**4 points**] Determine the equation of the tangent line to $y = \frac{1}{x^2}$ at the point where $x = 1$. You result from part (a) should be useful here.

If $x=1$, $y = \frac{1}{1^2} = 1$, so point is $(1,1)$.

Slope of tangent line is $f'(1) = \frac{-2}{1^3} = -2$.

$\therefore$ Equation of tangent line is $y-1 = -2(x-1)$

<u>or</u> $y = -2x+3$