 Name: SOLUTIONS

Question 1: Find the derivatives of the following functions (you do not have to simplify your answers):

(a)[3 points] $f(x) = -2x^5 + 5\pi^2 - \frac{1}{x^2}$ $= -2x^5 + 5\pi^2 - x^{-2}$

$f'(x) = -10x^4 + 2x^{-3}$

(b)[3 points] $y = \sqrt{\frac{1}{t^2 - 7}}$ $= (t^2-7)^{-\frac{1}{2}}$

$y' = -\frac{1}{2}(t^2-7)^{-3/2}(2t)$

(c)[4 points] $g(x) = 3x - 5\cos^2(\pi x)$ $= 3x - 5[\cos(\pi x)]^2$

$g'(x) = 3 - 10[\cos(\pi x)] \cdot (-\sin(\pi x)) \cdot \pi$

Question 2:

(a)[3 points] Find $f'(x)$ if $f(x) = \dfrac{\sqrt{x+1}}{x^2+1}$. $= \dfrac{(x+1)^{1/2}}{x^2+1}$

$$f'(x) = \frac{(x^2+1)\frac{1}{2}(x+1)^{-1/2} - (x+1)^{1/2}(2x)}{(x^2+1)^2}$$

(b)[3 points] Find y' if $y = \cos(\sin(x^2+x))$

$$y' = -\sin(\sin(x^2+x)) \cdot \cos(x^2+x) \cdot (2x+1)$$

(c)[4 points] y is defined implicitly as a function of x by the equation $x \sin y = y \cos x$. Find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}[x \sin y] = \frac{d}{dx}[y \cos x]$$

$$1 \cdot \sin y + x \cos y \cdot y' = y' \cos x - y \sin x$$

$$y'[x \cos y - \cos x] = -y \sin x - \sin y$$

$$y' = -\frac{y \sin x + \sin y}{x \cos y - \cos x}.$$

Question 3:

(a)[3 points] Compute $\int \frac{\sqrt{x}}{2} - 3\sin x\, dx$. $= \int \frac{1}{2} x^{1/2} - 3\sin x \, dx$

$$= \frac{1}{2} \frac{x^{3/2}}{\frac{3}{2}} + 3\cos x + C$$

$$= \frac{1}{3} x^{3/2} + 3\cos x + C.$$

(b)[3 points] A bug walking along the x-axis has position given as a function of time $x(t) = 2t^3 - 15t^2 + 24t$, where t is in seconds. What is the acceleration of the bug at time $t = 2$ seconds?

$$x'(t) = 6t^2 - 30t + 24$$

$$x''(t) = 12t - 30$$

$$x''(2) = 12(2) - 30 = -6$$

(c)[4 points] Solve the differential equation $y' = t^{3/2} - t - 1$ where $y = 0$ when $t = 0$.

$$y = \int t^{3/2} - t - 1 \, dt$$

$$y = \frac{2}{5} t^{5/2} - \frac{t^2}{2} - t + C$$

$$y(0) = 0 \Rightarrow C = 0$$

$$\therefore\ y = \frac{2}{5} t^{5/2} - \frac{t^2}{2} - t$$

Question 4:

(a)[5 points] Find the equation of the tangent line to the curve $y = \dfrac{1+\sin x}{1-\sin x}$ at the point where $x = \pi$.

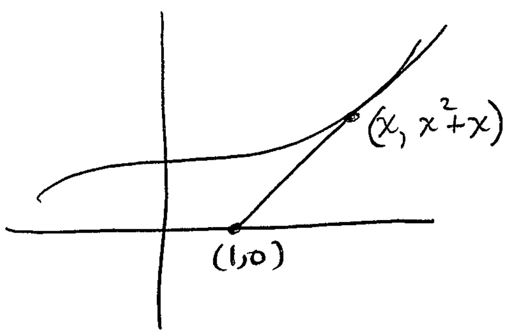
$$y\Big|_{x=\pi} = \frac{1+\sin\pi}{1-\sin\pi} = 1$$

$$y' = \frac{(1-\sin x)(\cos x) + (1+\sin x)(\cos x)}{(1-\sin x)^2}$$

$$y'\Big|_{x=\pi} = \frac{(1)(-1)+(1)(-1)}{1^2} = -2$$

$$\therefore \quad y - 1 = -2(x-\pi).$$

(b)[5 points] There are two points on the curve $y = x^2 + x$ at which the tangent lines to the curve also pass through $(1, 0)$. Find the x coordinates of these points.



$$y' = 2x+1$$

$$\therefore \quad 2x+1 = \frac{x^2+x-0}{x-1}$$

$$(2x+1)(x-1) = x^2+x$$

$$2x^2+\cancel{x}-2x-1 = x^2+\cancel{x}$$

$$x^2-2x-1 = 0$$

$$x = \frac{2 \pm \sqrt{4-4(1)(-1)}}{2}$$

$$= \frac{2\pm\sqrt{8}}{2} = \frac{\cancel{2}\pm\cancel{2}\sqrt{2}}{\cancel{2}} = 1\pm\sqrt{2}.$$

Name: SOLUTIONS

Question 5:

(a)[**7 points**] Let $f(x) = (1+x)\sin(x)$. Use a tangent line approximation to estimate $f(0.1)$. Round your answer to 2 decimal places.

$$f(0) = 0$$

$$f'(x) = \sin(x) + (1+x)\cos(x)$$

$$f'(0) = 1$$

$$\therefore \quad f(x) \approx f(0) + f'(0)(x-0)$$

$$f(0.1) \cong 0 + (1)(0.1)$$

$$= 0.1$$

(b)[**3 points**] Is your estimate in part (a) an over-estimate or under-estimate? Explain using calculus.

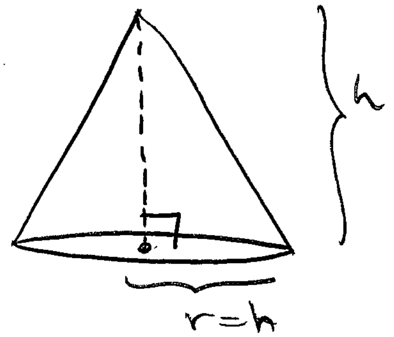
$$f''(x) = \cos(x) + \cos(x) + (1+x)(-\sin x)$$

$$f''(0) = 1 + 1 + 0 = 2 > 0 \Rightarrow$$

$\therefore$ under-estimate.

 Name: SOLUTIONS

Question 6: [10 points]
Sand is falling onto a pile at a rate of 1 m^3 per minute. The pile of sand maintains the shape of a cone with height equal to half the diameter of the base. How fast is the height of the sand pile increasing when the height is exactly 3 m? Give units in your answer.



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3$$

$$\frac{dV}{dt} = 1 \ m^3/min$$

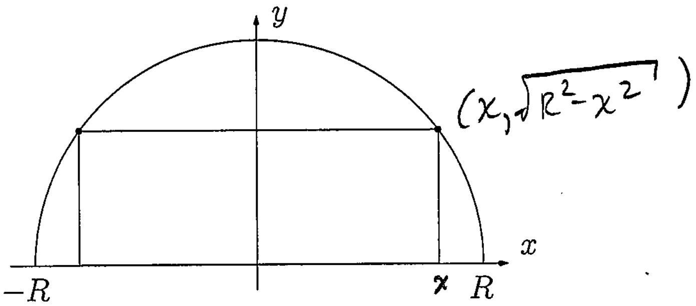
$$\frac{dV}{dt} = \frac{1}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$$

$$= \pi h^2 \frac{dh}{dt}$$

At $h=3$: $$1 = \pi \cdot 9 \cdot \frac{dh}{dt}$$

$$\boxed{\therefore \frac{dh}{dt} = \frac{1}{9\pi} \ m/min}$$

Question 7: [10 points]
Find the area of the largest rectangle that can be inscribed in a semicircle of radius R if one side of the rectangle lies along the diameter of the semicircle.



$$A(x) = 2x\sqrt{R^2-x^2}, \quad 0 \le x \le R.$$

$$A'(x) = 2\sqrt{R^2-x^2} + \cancel{2}x \, \frac{1}{\cancel{2}} (R^2-x^2)^{-\frac{1}{2}}(-2x)$$

$$= 2(R^2-x^2)^{\frac{1}{2}} - \frac{2x^2}{(R^2-x^2)^{1/2}}$$

$$= \frac{2(R^2-x^2) - 2x^2}{(R^2-x^2)^{1/2}}$$

$$= \frac{2R^2 - 2x^2 - 2x^2}{(R^2-x^2)^{1/2}}$$

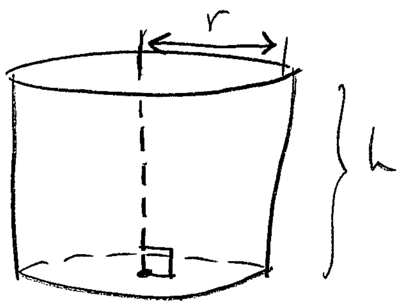
$$= \frac{2(R^2 - 2x^2)}{(R^2-x^2)^{1/2}}$$

$$= 0 \quad \text{when } R^2 = 2x^2 \Rightarrow x = \frac{R}{\sqrt{2}}$$

x	$A(x)$
0	0
$\frac{R}{\sqrt{2}}$	$2\frac{R}{\sqrt{2}}\sqrt{R^2 - \frac{R^2}{2}} = R^2$
R	0

$\boxed{\therefore \text{ maximum area is } R^2.}$

Question 8: [10 points]
A cylindrical container open at the top is to be made with 300π m^2 of material. Find the radius and height which produce the container of greatest volume. Be sure to justify that your answer does indeed give the maximum.



$$S = \cancel{\pi}r^2 + 2\cancel{\pi}rh = 300\cancel{\pi}$$

$$h = \frac{300 - r^2}{2r}$$

$$V = \pi r^2 h = \pi r^{\cancel{2}}\left(\frac{300 - r^2}{2\cancel{r}}\right)$$

$$= \frac{\pi}{2} r(300 - r^2), \quad 0 < r \le \sqrt{300}$$

$$V'(r) = \frac{\pi}{2}(300 - r^2) + \frac{\pi}{2}r(-2r)$$

$$= \frac{\pi}{2}\left[300 - r^2 - 2r^2\right]$$

$$= \frac{\pi}{2}\left[300 - 3r^2\right]$$

$V'(r) = 0$ at $r = 10$

$V''(r) = \frac{\pi}{2}(-6r) < 0$ for any $r > 0$,

$\therefore$ $r = 10$ gives abs. max.

$\therefore \boxed{r = 10 \text{ m},}$ $\quad h = \frac{300 - r^2}{2r} = \frac{300 - 100}{20} = \frac{20\cancel{0}}{2\cancel{0}}$

$\boxed{\therefore h = 10 \text{ m}}$

Name: SOLUTIONS

Question 9:

(a)[5 points] You are using Newton's Method to locate a root of the equation $f(x) = 0$, and you make an initial guess of $x_1 = 2$ for the root. The tangent line to the curve $y = f(x)$ at $x = 2$ has equation $3y = 10x - 19$. What will be the next approximation x_2 obtained using Newton's Method?

$$y = \frac{10}{3}x - \frac{19}{3}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{\left(\frac{20}{3} - \frac{19}{3}\right)}{\frac{10}{3}}$$

$$= \cancel{2} - \cancel{\frac{20}{3} \cdot \frac{3}{10}} + \frac{19}{3} \cdot \frac{3}{10}$$

$$= \boxed{\frac{19}{10}}$$

or $= 1.9$

(b)[5 points] The equation $x^2 = \dfrac{1}{\sin x}$ has one solution for $0 < x < 2$. Find x_2, the second approximation for the root obtained using Newton's Method. Round your answer to three decimal places.

$$f(x) = x^2 \sin x - 1 = 0$$

$$f'(x) = 2x\sin x + x^2 \cos x$$

$$\therefore x_{n+1} = x_n - \frac{(x_n)^2 \sin(x_n) - 1}{2(x_n)\sin(x_n) + (x_n)^2\cos(x_n)}$$

x	$f(x)$
0	-1
1	-0.1585
2	0.8

$\rightarrow$ use $x_1 = \frac{1+2}{2} = 1.5$

n	x_n	x_{n+1}
1	1.5	1.105

$\boxed{\therefore x_2 = 1.105}$

Math 121 Sec I06N01 - Final Exam
June 15 2006 Name: SOLUTIONS

Question 10: Consider the function $f(x) = \dfrac{x^3}{1+x}$. Note that this function has a vertical asymptote at $x = -1$.

(a)[**2 points**] Find the x and y intercepts of the graph of $y = f(x)$.

x-int: $\dfrac{x^3}{1+x} = 0$ at $x = 0$; $\therefore (0,0)$

y-int: $f(0) = 0$: $(0,0)$.

(b)[**3 points**] Find the intervals of increase and decrease of $f(x)$. State the x coordinate of any relative extrema.

$$f'(x) = \frac{(1+x)3x^2 - x^3}{(1+x)^2} = \frac{2x^3 + 3x^2}{(1+x)^2} = \frac{x^2(2x+3)}{(1+x)^2}$$

$f'(x) = 0$ at $x = 0, -\frac{3}{2}$

$f'(x)$ not defined at $x = -1$ (f not continuous at $x = -1$).

	-2	$-\frac{3}{2}$	-1.25	-1	$-\frac{1}{2}$	0	1
$f'(x)$:	$-$		$+$		$+$		$+$
$f(x)$:	$\searrow$		$\nearrow$		$\nearrow$		$\nearrow$

$\therefore$ increasing on $(-\frac{3}{2}, -1), (-1, 0), (0, \infty)$; decreasing on $(-\infty, -\frac{3}{2})$; rel. min at $x = -\frac{3}{2}$.

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(c)[3 points] Find the intervals on which $f(x)$ is concave up, and the intervals on which $f(x)$ is concave down. State the x coordinate of any inflection points.

$$f''(x) = \frac{(1+x)^{\cancel{2}}(6x^2+6x) - (2x^3+3x^2)\,2\cancel{(1+x)}}{(1+x)^{\cancel{4}\,3}}$$

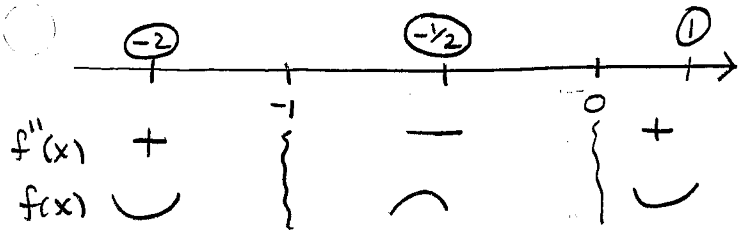
$$= \frac{6x^2+6x^3+6x+\cancel{6x^2}-4x^3-\cancel{6x^2}}{(1+x)^3}$$

$$= \frac{2x^3+6x^2+6x}{(1+x)^3}$$

$$= \frac{2x(x^2+3x+3)}{(1+x)^3}$$

$f''(x) = 0$ at $x = 0$

$f''(x)$ not defined at $x = -1$



$\therefore$ f(x) concave up on $(-\infty,-1)$, $(0,\infty)$;
concave down on $(-1,0)$
inflection point at $x=0$.

(d)[2 points] Use your results from (a), (b) and (c) to sketch the graph of $y = f(x)$.

x	y
0	0
$-\frac{3}{2}$	6.75