**Question 1:** Find the derivatives of the following functions (you do not have to simplify your answers):

**(a)[3 points]** $y = \sqrt{x} - \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}$ $= x^{\frac{1}{2}} - x^{-\frac{1}{2}} - \frac{1}{\sqrt{2}}$

$y' = \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}$

**(b)[3 points]** $f(x) = x^5 \sin x$

$f'(x) = 5x^4 \sin x + x^5 \cos x$

**(c)[4 points]** $g(t) = \frac{t^3}{(t-1)^3}$

$g'(t) = \frac{(t-1)^3(3t^2) - (t^3)\,3(t-1)^2}{(t-1)^6}$

**Question 2:**

(a)[**3 points**] Let $f(x) = \dfrac{\sin(\pi x)}{x+2}$. Compute $f'(2)$.

$$f'(x) = \frac{(x+2)\cos(\pi x)\cdot\pi - \sin(\pi x)(1)}{(x+2)^2}$$

$$\therefore \quad f'(2) = \frac{(4)\cos(\cancel{2\pi})^{1}\cdot\pi - \cancel{\sin(2\pi)}^{0}}{4^2}$$

$$= \frac{\pi}{4}$$

(b)[**3 points**] Find $\dfrac{dy}{dx}$ where $y = \left(x^2 + \dfrac{1}{x}\right)^5$

$$\frac{dy}{dx} = 5\left(x^2 + \frac{1}{x}\right)^4 \left(2x - \frac{1}{x^2}\right)$$

(c)[**4 points**] Compute $y'$, where $y = \cos^2(\sqrt{x^4+1})$. $= \left[\cos\left(\sqrt{x^4+1}\right)\right]^2$

$$y' = 2\cos\left(\sqrt{x^4+1}\right)\left(-\sin\left(\sqrt{x^4+1}\right)\left(\tfrac{1}{2}\right)\left(x^4+1\right)^{-\frac{1}{2}}\left(4x^3\right)\right.$$

**Question 3:**

(a)[**3 points**] Compute $\lim_{x\to 9^-} \frac{3-\sqrt{x}}{x-9}$.

$$\lim_{x\to 9^-} \frac{3-\sqrt{x}}{x-9} = \lim_{x\to 9^-} \frac{3-\sqrt{x}}{x-9} \cdot \frac{3+\sqrt{x}}{3+\sqrt{x}}$$

$$= \lim_{x\to 9^-} \frac{9-x}{(x-9)(3+\sqrt{x})}$$

$$= \lim_{x\to 9^-} \frac{\cancel{-(x-9)}}{\cancel{(x-9)}(3+\sqrt{x})} = \frac{-1}{6}.$$

(b)[**3 points**] Compute $\int \frac{x^2+2x-3}{x^4}\,dx$.

$$\int \frac{x^2+2x-3}{x^4}dx = \int x^{-2} + 2x^{-3} - 3x^{-4}\,dx$$

$$= -x^{-1} - \frac{2x^{-2}}{2} + \frac{3x^{-3}}{3} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C.$$

(c)[**4 points**] The function $f(x)$ satisfies the differential equation $f''(x) = 2x+1$, where $f(0)=1$ and $f'(0)=-1$. Find a formula for $f(x)$.

$$f'(x) = \int 2x+1\,dx = \frac{2x^2}{2} + x + C = x^2+x+C$$

$$f'(0) = -1 \Rightarrow 0+0+C = -1 \Rightarrow C=-1$$

$$\therefore f'(x) = x^2+x-1$$

$$\therefore f(x) = \int x^2+x-1\,dx = \frac{x^3}{3} + \frac{x^2}{2} - x + C$$

$$f(0)=1 \Rightarrow 0+0-0+C = 1$$

$$\Rightarrow C=1$$

$$\boxed{\therefore f(x) = \frac{x^3}{3} + \frac{x^2}{2} - x + 1}$$

**Question 4:** $y$ is defined implicitly as a function of $x$ by the equation

$$y^2(x^2+y^2) = 2x^2 .$$

**(a)[7 points]** Find the equation of the tangent line to the the graph of this equation at the point $(1,1)$.

$$2yy'(x^2+y^2) + y^2(2x+2yy') = 4x$$

at $(1,1)$: $2(1)y'(1+1) + (1)(2+2(1)y') = 4(1)$

$$4y' + 2 + 2y' = 4$$

$$6y' = 2$$

$$y' = \frac{1}{3}$$

$\therefore \quad y-1 = \frac{1}{3}(x-1).$

**(b)[3 points]** If the point $(1.1, a)$ is on the graph of $y^2(x^2+y^2) = 2x^2$, use your result from part (a) to estimate the value of $a$.

$$y-1 = \frac{1}{3}(x-1)$$

$$y = \frac{1}{3}(x-1)+1$$

$\therefore$ When $x = 1.1$, $a \approx \frac{1}{3}(1.1-1)+1 = \frac{31}{30}$

$\therefore \quad a \approx \frac{31}{30} \doteq 1.0\overline{3}$

**Question 5:** [10 points]
An oil tanker ship hits a reef and begins leaking oil at the rate of 100 $m^3$ per hour. The oil slick forms a large disc of uniform thickness 0.03 m (that is, the oil slick is a very wide and short cylinder floating on the surface of the water). How fast is the radius of the disc growing two hours after the tanker hit the reef? Round your answer to 1 decimal, and state units with your answer.


$$\frac{dV}{dt} = 100 \ \frac{m^3}{hr.}$$

At $t = 2$ hrs, $V = 200 \ m^3$,

$\therefore$ Find $\frac{dr}{dt}$ when $V = 200$.

$$V = \pi r^2 h$$

$$\therefore \ r = \left[\frac{1}{\pi h} V\right]^{\frac{1}{2}} = \left[\frac{1}{\pi (0.03)}\right]^{1/2} V^{1/2}$$

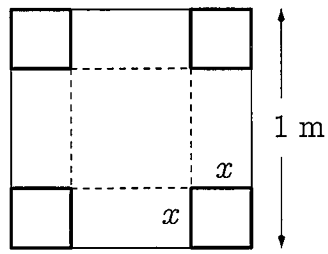
$$\therefore \ \frac{dr}{dt} = \left[\frac{1}{\pi (0.03)}\right]^{1/2} \frac{1}{2} V^{-\frac{1}{2}} \frac{dV}{dt}$$

When $V = 200$: $$\frac{dr}{dt} = \left[\frac{1}{\pi (0.03)}\right]^{\frac{1}{2}} \left(\frac{1}{2}\right) (200)^{-\frac{1}{2}} (100)$$

$$\boxed{\therefore \ \frac{dr}{dt} \doteq 11.5 \ \frac{m}{hr}}$$

**Question 6:** [10 points]

Three large square pieces of tin of side length 1 m each have four squares of side length $x$ cut from their corners. The twelve cut out squares are used to form two closed cubes. The three cross shaped pieces of tin are folded to make three boxes without tops.



one of the three pieces of tin

**(a)** [**2 points**] Let $V(x)$ be the total volume of the 2 cubes and 3 boxes. Determine a formula for $V(x)$.

$$V(x) = 2x^3 + 3x(1-2x)^2$$

**(b)** [**1 point**] Establish the domain of $V(x)$. Have your domain include the possibilities that (i) no small squares are cut out (i.e. $x = 0$), and (ii) the other extreme: the twelve cut out squares use up all of the tin.

Domain: $[0, \frac{1}{2}]$.

**(c)** [**7 points**] Using (a) and (b), determine the value of $x$ which produces the maximum possible total volume of the boxes. Round your answers to 2 decimals.

maximize $V(x) = 2x^3 + 3x(1-2x)^2$ on $[0, \frac{1}{2}]$:

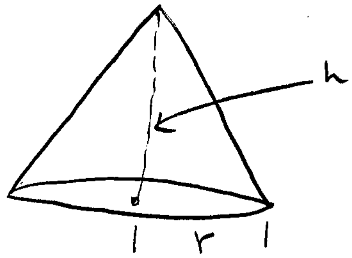
$$V'(x) = 6x^2 + 3(1-2x)^2 + 3x(2)(1-2x)(-2)$$
$$= 6x^2 + 3(1-4x+4x^2) - 12x + 24x^2$$
$$= 42x^2 - 24x + 3$$
$$= 3(14x^2 - 8x + 1).$$

$$V'(x) = 0 \Rightarrow x = \frac{8 \pm \sqrt{64 - 4(14)(1)}}{28}$$
$$= \frac{8 \pm \sqrt{8}}{28} = \frac{8 \pm 2\sqrt{2}}{28}$$
$$= \frac{4 \pm \sqrt{2}}{14}$$

| $x$ | $V(x)$ |
|---|---|
| 0 | 0 |
| $\frac{4+\sqrt{2}}{14}$ | 0.18 |
| $\frac{4-\sqrt{2}}{14}$ | 0.23 |
| $\frac{1}{2}$ | 0.25 |

$\therefore$ $x = \frac{1}{2}$ produces maximum volume

**Question 7: [10 points]**
We wish to construct a cone of volume $\pi/3$ m$^3$ in such a way that the total of the height and circumference of the circular base is a minimum. Find the dimensions of the cone. (Be sure to justify that you have indeed found the minimum.)




$$V = \frac{\pi}{3} = \frac{1}{3}\pi r^2 h$$

Let $L = h + 2\pi r$

$\therefore$ minimize $L = h + 2\pi r$ ①
subject to $\frac{\pi}{3} = \frac{1}{3}\pi r^2 h$. ②.

Using ② : $1 = r^2 h$, so $h = \frac{1}{r^2}$.

$\therefore$ ① becomes $L(r) = 2\pi r + \frac{1}{r^2}$, $r > 0$

$$L'(r) = 2\pi - \frac{2}{r^3} = 2\left(\frac{\pi r^3 - 1}{r^3}\right).$$

$$L'(r) = 0 \Rightarrow r^3 = \frac{1}{\pi}$$

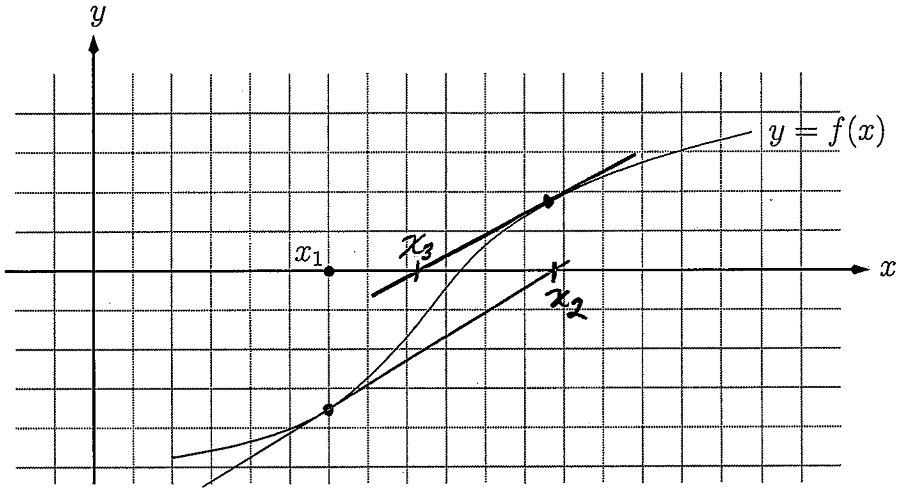
$$\therefore r = \left(\frac{1}{\pi}\right)^{1/3}$$

Since $L''(r) = \frac{6}{r^4} > 0$ for all $r > 0$,
$r = \left(\frac{1}{\pi}\right)^{1/3}$ gives a relative & absolute minimum of $L$.

$\therefore$ $r = \left(\frac{1}{\pi}\right)^{1/3}$ m, and $h = \frac{1}{r^2} = \frac{1}{\left(\left(\frac{1}{\pi}\right)^{1/3}\right)^2} = \pi^{2/3}$ m.

**Question 8:** [10 points]

**(a)** [**3 points**] We wish to find the $x$ intercept of the graph below using Newton's Method. Beginning with the starting value $x_1$ indicated, use a straight-edge to draw tangent lines to locate $x_2$ and $x_3$, the next two approximations given by Newton's Method. Clearly indicate and label the points corresponding to $x_2$ and $x_3$.




**(b)** [**7 points**] You wish to solve the equation $x = \cos(2x)$ using Newton's Method. Given that the graphs of $y = x$ and $y = \cos(2x)$ have one point of intersection between $x = 0$ and $x = 1$, find $x_1$, $x_2$ and $x_3$, the first three approximations to the solution of the equation using Newton's Method. Round your calculated values to 4 decimal places.

Solve $f(x) = x - \cos(2x) = 0$

$\therefore f'(x) = 1 + 2\sin(2x)$

$\therefore X_{n+1} = X_n - \dfrac{f(x_n)}{f'(x_n)}$

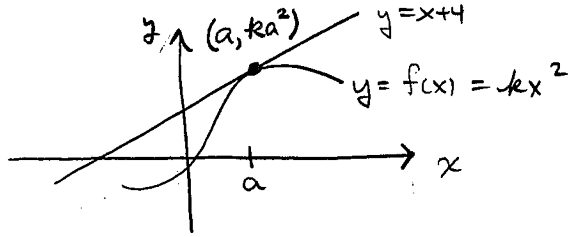
$$X_{n+1} = X_n - \frac{x_n - \cos(2x_n)}{1 + 2\sin(2x_n)}$$

$x_1 = \frac{0+1}{2} = 0.5$

| $n$ | $x_n$ |
|---|---|
| 1 | 0.5 |
| 2 | 0.5150 |
| 3 | 0.5149 |

**Question 9:**

**(a)** [**5 points**] Suppose the line $y = x + 4$ is tangent to the graph of $f(x) = kx^2$, where $k$ is some constant. What must be the value of $k$?




$f'(x) = 2kx$ ; $f'(a) = 2ka$

At $x = a$ we have :

① $a + 4 = ka^2$

② $1 = 2ka$

using ② : $k = \frac{1}{2a}$

sub-into ① : $a+4 = \left(\frac{1}{2a}\right)a^2$

$a + 4 = \frac{a}{2}$

$2a + 8 = a$

$a = -8$

$\therefore k = \frac{1}{2a} = \boxed{\frac{-1}{16}}$

**(b)** [**5 points**] The displacement of an object is given by $s(t) = \dfrac{3t}{2t^2+1}$, where $t > 0$ is time. When is the object stationary?

When $s'(t) = 0$ : $s'(t) = \dfrac{(2t^2+1)(3) - (3t)(4t)}{(2t^2+1)^2}$

$= \dfrac{6t^2 + 3 - 12t^2}{(2t^2+1)^2}$

$= \dfrac{3 - 6t^2}{(2t^2+1)^2}$

$= \dfrac{3(1-2t^2)}{(2t^2+1)^2}$

$\therefore s'(t) = 0 \Rightarrow 1 - 2t^2 = 0$

$t^2 = \frac{1}{2}$

$\boxed{t = \frac{1}{\sqrt{2}}}$

**Question 10:** Consider the function $f(x) = \dfrac{x^2+3x+4}{x^2+4}$.

**(a)** [**2 points**] Find the horizontal asymptote(s) as well as the $y$ intercept of the graph of $y = f(x)$.

$$\lim_{x\to\infty} \frac{x^2+3x+4}{x^2+4} = \lim_{x\to\infty} \frac{1+\frac{3}{x}+\frac{4}{x^2}}{1+\frac{4}{x^2}} = 1 \Big\} \therefore y=1 \text{ is a horiz. asymptote.}$$

$$\lim_{x\to-\infty} \frac{x^2+3x+4}{x^2+4} = \lim_{x\to-\infty} \frac{1+\frac{3}{x}+\frac{4}{x^2}}{1+\frac{4}{x^2}} = 1 \Big\} \text{ again,}$$

$f(0) = \frac{4}{4} = 1 \quad \therefore$ $y$-intercept at $(0,1)$

**(b)** [**3 points**] Find the intervals of increase and decrease of $f(x)$. State the $x$ and $y$ coordinates of any relative extrema.

$$f'(x) = \frac{(x^2+4)(2x+3) - (x^2+3x+4)(2x)}{(x^2+4)^2}$$

$$= \frac{\cancel{2x^3}+\cancel{8x}+3x^2+12-\cancel{2x^3}-6x^2-\cancel{8x}}{(x^2+4)^2}$$

$$= \frac{3(4-x^2)}{(x^2+4)^2}$$

$\therefore f(x) = 0$ at $x = 2, -2$:

| | $-3$ | $-2$ | $0$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f'(x)$: | $-$ | | $+$ | | $-$ |
| $f(x)$: | $\searrow$ | | $\nearrow$ | | $\searrow$ |

rel. min. at $(-2, \frac{1}{4})$,

rel. max. at $(2, \frac{7}{4})$

$\therefore f(x)$ increasing on $(-2, 2)$,

decreasing on $(-\infty, -2)$ and $(2, \infty)$

continued on next page...

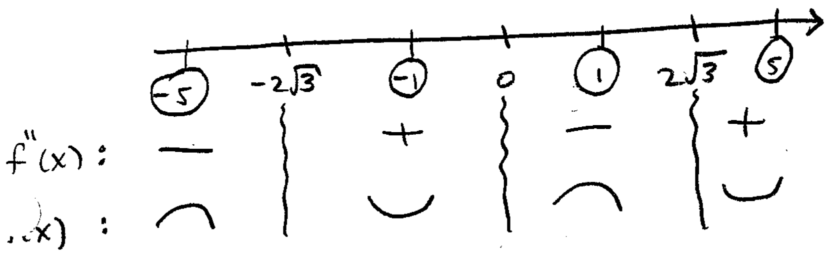
(c)[**3 points**] Find the intervals on which $f(x)$ is concave up, and the intervals on which $f(x)$ is concave down. State the $x$ and $y$ coordinates of any inflection points (you may wish to approximate the $y$-coordinates using your calculator).

$$f''(x) = \frac{(x^2+4)^2(-6x) - (12-3x^2)(2)(x^2+4)(2x)}{(x^2+4)^2}$$

$$= \frac{2x\cancel{(x^2+4)}\left[(x^2+4)(-3) - 2(12-3x^2)\right]}{(x^2+4)^{\cancel{2}}}$$

$$= \frac{2x\,(3x^2-36)}{(x^2+4)} = \frac{6x\,(x^2-12)}{x^2+4}$$

$$f''(x) = 0 \Rightarrow x = 0, \quad x = \pm\sqrt{12} = \pm 2\sqrt{3}$$




∴ concave down on $(-\infty, -2\sqrt{3})$ and $(0, 2\sqrt{3})$; concave up on $(-2\sqrt{3}, 0)$, $(2\sqrt{3}, \infty)$.

∴ inflection points at $(-2\sqrt{3}, 0.35)$, $(0, 1)$, $(2\sqrt{3}, 1.65)$,

(d)[**2 points**] Use your results from (a), (b) and (c) to make an informative sketch the graph of $y = f(x)$.

| $x$ | $f(x)$ |
|---|---|
| $-2\sqrt{3} \doteq -3.5$ | 0.35 |
| $-2$ | 0.25 |
| 0 | 1 |
| 2 | 1.75 |
| $2\sqrt{3} \doteq 3.5$ | 1.65 |