

Question 1:

(a)[3 points] Evaluate $\int \frac{\tan^{-1} x}{1+x^2} dx$.

$$u = \tan^{-1} x$$
$$du = \frac{1}{1+x^2} dx$$

$$\therefore \int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{u^2}{2} + C = \boxed{\frac{[\tan^{-1} x]^2}{2} + C}$$

(b)[3 points] Suppose $\int_0^5 f'(x) dx = 11$, where $f'(x)$ is continuous. If $f(0) = -2$, what is $f(5)$?

$$11 = \int_0^5 f'(x) dx$$

$$= f(5) - f(0)$$

$$= f(5) - (-2)$$

$$\therefore f(5) = 11 - 2 = \boxed{9}$$

(c)[4 points] Suppose the average value of $f(x) = 6x(x-1)$ over the interval $x = 0$ to $x = k$ is k . What is k ?

$$k = \frac{1}{k-0} \int_0^k 6x(x-1) dx$$

$$= \frac{1}{k} \int_0^k 6x^2 - 6x dx$$

$$= \frac{1}{k} \left[\frac{6x^3}{3} - \frac{6x^2}{2} \right]_0^k$$

$$= \frac{1}{k} [2k^3 - 3k^2]$$

$$= 2k^2 - 3k$$

$$\therefore 2k^2 = 4k$$

$$\boxed{k = 2}$$

Question 2 [10 points]: Evaluate $\frac{9}{4} \int_1^4 \sqrt{t} \ln t \, dt$.

$$I = \int t^{1/2} \ln t \, dt \quad u = \ln t \quad dv = t^{1/2} \, dt$$
$$du = \frac{1}{t} \, dt \quad v = \frac{2}{3} t^{3/2}$$

$$\begin{aligned} \therefore I &= \frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \int t^{3/2} \frac{1}{t} \, dt \\ &= \frac{2}{3} t^{3/2} \ln t - \frac{2}{3} \int t^{1/2} \, dt \\ &= \frac{2}{3} t^{3/2} \ln t - \frac{4}{9} t^{3/2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{9}{4} \int_1^4 \sqrt{t} \ln t \, dt &= \frac{9}{4} \left[\frac{2}{3} t^{3/2} \ln t - \frac{4}{9} t^{3/2} \right]_1^4 \\ &= \frac{9}{4} \left[\left(\frac{2}{3} \cdot 4^{3/2} \cdot \ln 4 - \frac{4}{9} \cdot 4^{3/2} \right) - \left(\frac{2}{3} \cdot 1 \cdot \ln 1 - \frac{4}{9} \cdot 1 \right) \right] \\ &= \frac{9}{4} \left[\frac{16}{3} \ln 4 - \frac{32}{9} + \frac{4}{9} \right] \\ &= \boxed{12 \ln 4 - 7} \end{aligned}$$

Question 3 [10 points]: Evaluate

$$I = \int \frac{x^2}{16\sqrt{16-x^2}} dx$$

$$x = 4 \sin \theta$$

$$dx = 4 \cos \theta d\theta$$

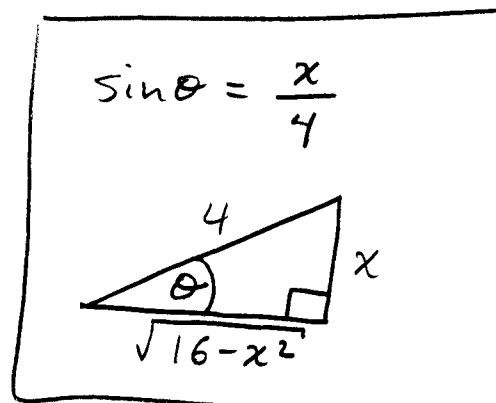
$$\therefore I = \int \frac{16 \sin^2 \theta \cdot 4 \cos \theta d\theta}{16 \sqrt{16 - 16 \sin^2 \theta}}$$

$$= \int \frac{\sin^2 \theta \cdot 4 \cancel{\cos \theta}}{4 \cancel{\cos \theta}} d\theta$$

$$= \int \frac{1}{2} - \frac{\cos(2\theta)}{2} d\theta$$

$$= \frac{\theta}{2} - \frac{1}{2} \frac{\sin(2\theta)}{2} + C$$

$$= \frac{\theta}{2} - \frac{1}{2} \sin \theta \cos \theta + C$$



$$= \frac{1}{2} \arcsin\left(\frac{x}{4}\right) - \frac{1}{2} \frac{x}{4} \frac{\sqrt{16-x^2}}{4} + C$$

$$= \frac{1}{2} \arcsin\left(\frac{x}{4}\right) - \frac{1}{32} x \sqrt{16-x^2} + C$$

Question 4 [10 points]: Evaluate

$$\int \frac{3x^2 + 8}{x^3 + 4x} dx$$

$$\begin{aligned} \frac{3x^2 + 8}{x^3 + 4x} &= \frac{3x^2 + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \\ &= \frac{Ax^2 + 4A + Bx^2 + Cx}{x(x^2 + 4)} \\ &= \frac{(A+B)x^2 + Cx + 4A}{x(x^2 + 4)} \end{aligned}$$

$$\therefore A + B = 3$$

$$C = 0$$

$$4A = 8 \rightarrow \therefore A = 2 \quad ; \quad \therefore B = 3 - A = 3 - 2 = 1$$

$$\therefore I = \int \frac{2}{x} + \frac{x}{x^2 + 4} dx$$

$$= 2 \int \frac{1}{x} dx + \underbrace{\left(\frac{1}{2} \right) \int \frac{(2)x}{x^2 + 4} dx}_{\substack{u = x^2 + 4 \\ du = 2x dx}}$$

$$= 2 \ln|x| + \frac{1}{2} \int \frac{1}{u} du$$

$$= 2 \ln|x| + \frac{1}{2} \ln|u| + C$$

$$= \boxed{2 \ln|x| + \frac{1}{2} \ln|x^2 + 4| + C}$$

Question 5 [10 points]: Evaluate

$$\int_0^{\pi/3} \sin^7(3t) \cos^3(3t) dt$$

$$I = \int \sin^7(3t) \cos^3(3t) dt$$

$$= \left(\frac{1}{3}\right) \int \sin^7(3t) (1 - \sin^2(3t)) \cos(3t) dt$$

$$u = \sin(3t)$$

$$du = 3 \cos(3t) dt$$

$$\therefore I = \frac{1}{3} \int u^7 (1 - u^2) du$$

$$= \frac{1}{3} \int u^7 - u^9 du$$

$$= \frac{1}{3} \left[\frac{u^8}{8} - \frac{u^{10}}{10} \right]$$

$$= \frac{1}{3} \left[\frac{\sin^8(3t)}{8} - \frac{\sin^{10}(3t)}{10} \right]$$

$$\therefore \int_0^{\pi/3} \sin^7(3t) \cos^3(3t) dt$$

$$= \frac{1}{3} \left[\frac{\sin^8(3t)}{8} - \frac{\sin^{10}(3t)}{10} \right]_0^{\pi/3}$$

$$= \frac{1}{3} \left[\left(\frac{\sin^8(\pi)}{8} - \frac{\sin^{10}(\pi)}{10} \right) - \left(\frac{\sin^8(0)}{8} - \frac{\sin^{10}(0)}{10} \right) \right] = \boxed{0}$$