Question 1:

(a)[3 points] Find and simplify $f'(x)$ where

$$f(x) = \arcsin(e^x) + \arccos(e^x)$$

$$f'(x) = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x - \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

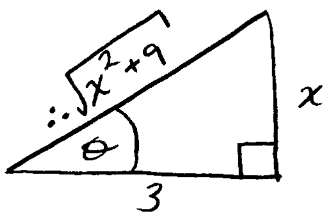
$$= \boxed{0}$$

(b)[3 points] Write as a simplified expression which does not involve trigonometric functions:

$$\cos\left[\tan^{-1}\left(\frac{x}{3}\right)\right]$$

let $\theta = \tan^{-1}\left(\frac{x}{3}\right)$

so $\tan\theta = \frac{x}{3}$:

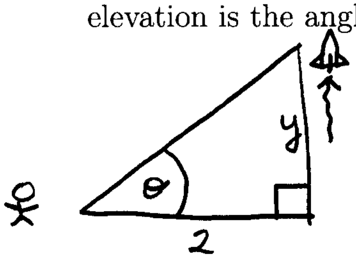


$$\therefore \cos\left[\tan^{-1}\left(\frac{x}{3}\right)\right]$$

$$= \cos\theta$$

$$= \boxed{\frac{3}{\sqrt{x^2+9}}}$$

(c)[4 points] An observer located 2 km from the launchpad watches a rocket launch. When the rocket is 1/5 km above the ground it is travelling at 200 km/h. How fast is the angle of elevation of the rocket changing at that instant? State units with your answer. (The angle of elevation is the angle between the ground and line of sight from observer to rocket.)



When $y = \frac{1}{5}$ km, $\frac{dy}{dt} = 200 \frac{km}{h}$

Find $\frac{d\theta}{dt}$ when $y = \frac{1}{5}$.

$$\theta = \arctan\left(\frac{y}{2}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{y}{2}\right)^2} \cdot \frac{1}{2} \cdot \frac{dy}{dt}$$

When $y = \frac{1}{5}$ km :

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1}{10}\right)^2} \cdot \frac{1}{2} \cdot 200$$

$$= \frac{1}{\frac{101}{100}} \cdot 100$$

$$= \boxed{\frac{10{,}000}{101} \ \frac{\text{radians}}{\text{hr}}}$$

Question 2:

(a)[5 points] Find all values of x at which the tangents to $f(x) = \sinh^2\left(\frac{x}{2}\right)$ have the same slope as tangents to $g(x) = \cosh\left(\frac{x}{2}\right)$.

Solve $f'(x) = g'(x)$

$$2\sinh\left(\tfrac{x}{2}\right)\cosh\left(\tfrac{x}{2}\right)\cdot\cancel{\left(\tfrac{1}{2}\right)} = \sinh\left(\tfrac{x}{2}\right)\cdot\cancel{\left(\tfrac{1}{2}\right)}$$

$$2\sinh\left(\tfrac{x}{2}\right)\cosh\left(\tfrac{x}{2}\right) - \sinh\left(\tfrac{x}{2}\right) = 0$$

$$\sinh\left(\tfrac{x}{2}\right)\left[2\cosh\left(\tfrac{x}{2}\right) - 1\right] = 0$$

$\therefore \sinh\left(\frac{x}{2}\right) = 0$, $2\cosh\left(\frac{x}{2}\right) - 1 = 0$

$\frac{x}{2} = 0$

$\boxed{x = 0}$

$\underbrace{\cosh\left(\frac{x}{2}\right) = \frac{1}{2}}$

no solutions since $\cosh(t) \geq 1$ for every t.

(b)[5 points] Find

$$L = \lim_{x\to\infty}\left(2\cosh\left[\ln\left(\frac{x}{3}\right)\right] - 2\sinh\left[\ln\left(\frac{x}{3}\right)\right]\right)$$

It is best to do this problem without L'Hospital's rule.

$$L = \lim_{x\to\infty}\left(\cancel{2}\,\frac{e^{\ln\left(\frac{x}{3}\right)} + e^{-\ln\left(\frac{x}{3}\right)}}{\cancel{2}} - \cancel{2}\,\frac{e^{\ln\left(\frac{x}{3}\right)} - e^{-\ln\left(\frac{x}{3}\right)}}{\cancel{2}}\right)$$

$$= \lim_{x\to\infty}\left(e^{\ln\left(\frac{x}{3}\right)} + e^{\ln\left(\frac{3}{x}\right)} - \left(e^{\ln\left(\frac{x}{3}\right)} - e^{\ln\left(\frac{3}{x}\right)}\right)\right)$$

$$= \lim_{x\to\infty}\left(\cancel{\frac{x}{3}} + \frac{3}{x} - \cancel{\frac{x}{3}} + \frac{3}{x}\right)$$

$$= \boxed{0}$$

Question 3:

(a)[5 points] Find

$$\lim_{x\to 0^+} \sin x \ln x \quad \Big\}\ \text{"}0\cdot(-\infty)\text{"}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \quad \Big\}\ \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{\cos x}{\sin^2 x}\right)}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} \quad \Big\}\ \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = \boxed{0}$$

(b)[5 points] Find

$$\lim_{x\to 0^+} (\cos x)^{1/x^2} \quad \Big\}\ \text{"}1^\infty\text{"}$$

$$= \lim_{x\to 0^+} e^{\frac{1}{x^2}\ln(\cos x)}$$

consider

$$\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2} \quad \Big\}\ \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{-\frac{\sin x}{\cos x}}{2x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{2x\cos x} \quad \Big\}\ \text{"}\frac{0}{0}\text{"}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{-\cos x}{2\cos x - 2x\sin x} = \frac{-1}{2}$$

$$\therefore \lim_{x\to 0^+} e^{\frac{1}{x^2}\ln(\cos x)} = \boxed{e^{-\frac{1}{2}}}$$

Question 4:

(a) [**5 points**] A car travelling at 30 m/s applies the brakes and accelerates at $a(t) = -10t - 5$ m/s^2, where $t = 0$ corresponds to the instant the brakes are applied (notice the acceleration is negative since the car is slowing down.) How long does it take the car to come to a stop?

$a(t) = -10t - 5$

$v(0) = 30 \ \frac{m}{s}$

Find t_1 at which $v(t_1) = 0$

$v(t) = -\frac{10t^2}{2} - 5t + C_1$

$v(0) = 30$, so $30 = 0 + 0 + C_1$
$C_1 = 30$

$\therefore v(t) = -5t^2 - 5t + 30$

Solve $-5t^2 - 5t + 30 = 0$
$t^2 + t - 6 = 0$
$(t+3)(t-2) = 0$

since $t \geq 0$

$\therefore \cancel{t=-3}$, $t = 2$

$\therefore \boxed{t = 2 \text{ s}}$

(b) [**5 points**] Find $f(x)$ if

$$f''(x) = 15\sqrt{x} + \sinh(x), \quad f'(0) = 1, f(0) = \pi$$

$f'(x) = 15 \frac{x^{3/2}}{(3/2)} + \cosh(x) + C_1$
$= 10x^{3/2} + \cosh(x) + C_1$

$f'(0) = 1$, so $1 = 0 + 1 + C_1$
$\therefore C_1 = 0$

$\therefore f'(x) = 10x^{3/2} + \cosh(x)$

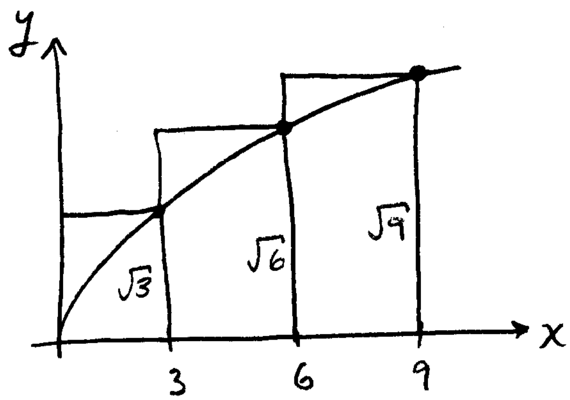
$f(x) = 10\frac{x^{5/2}}{(5/2)} + \sinh(x) + C_2$

$f(0) = \pi$, so $\pi = 0 + 0 + C_2$

$\boxed{\therefore f(x) = 4x^{5/2} + \sinh(x) + \pi}$

Question 5:

(a)[5 points] Use R_3 to estimate the area under the graph of $f(x) = \sqrt{x}$ over the interval $[0, 9]$. Round your answer to one decimal. Is your answer an over-estimate or under-estimate of the true area? Explain briefly.



$\therefore R_3 = (\sqrt{3})(3) + (\sqrt{6})(3) + (\sqrt{9})(3)$

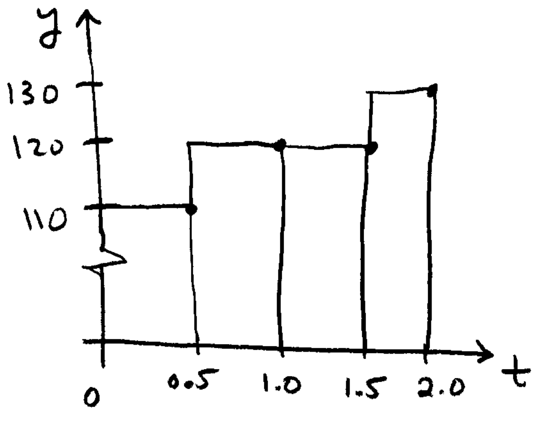
$\doteq \boxed{21.5}$

R_3 is an $\boxed{\text{over-estimate}}$ since $f(x) = \sqrt{x}$ is increasing, and so approximating rectangles extend above area under the curve of $f(x) = \sqrt{x}$.

(b)[5 points] The rate of increase of a growing town's population is determined at five points in time, resulting in the following data:

t (years)	0	0.5	1	1.5	2
$r(t)$ (people per year)	100	110	120	120	130

Give an upper estimate of the population increase over the period $t = 0$ to $t = 2$.



Increase in population

$\approx R_4$

$= (110)(0.5) + (120)(0.5) + (120)(0.5) + (130)(0.5)$

$= \boxed{240 \text{ people}}$