Math 122 Sec S08N03/04 - Quiz 5 Name:
Mar 10 2008 Stu#:

(1)[5 points] Approximate

$$I = \int_0^2 e^{-x^2}\,dx$$

using T_4, the trapezoid rule on four subintervals.

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$I \approx T_4 = \frac{(1/2)}{2}\left[e^{-0^2} + 2e^{-(\frac{1}{2})^2} + 2e^{-(1)^2} + 2e^{-(\frac{3}{2})^2} + e^{-2^2}\right]$$

$$= \frac{1}{4}\left[1 + 2e^{-\frac{1}{4}} + 2e^{-1} + 2e^{-\frac{9}{4}} + e^{-4}\right]$$

$$\doteq 0.8806$$

(2)[5 points] Give a bound on the error in your approximation in (a).

$$f(x) = e^{-x^2}$$

$$f'(x) = -2x e^{-x^2}$$

$$f''(x) = -2e^{-x^2} + (2x)^2 e^{-x^2} = 2e^{-x^2}(2x^2-1)$$

$$\therefore |f''(x)| = |2e^{-x^2}(2x^2-1)| \leq 2 \quad \text{on } [0,2] \quad \underbrace{(\text{Given}^*)}_{\text{See p.2}}$$

$$\therefore |E_T| \leq \frac{K(b-a)^3}{12n^2}$$

$$= \frac{\cancel{2}\cdot \cancel{2^3}}{12\cdot \cancel{4^2}}$$

$$= \frac{1}{12}$$

$$\doteq 0.0833$$

(3)[5 points] How large must n be for the approximation in (a) to be accurate to within 0.001?

Want $|E_T| < 0.001$;

enough to have

$$\frac{K(b-a)^3}{12n^2} < 0.001$$

$$\left[\frac{2 \cdot 2^3}{(12)(0.001)}\right]^{\frac{1}{2}} < n$$

$$\left[\frac{4000}{3}\right]^{\frac{1}{2}} < n$$

$$\therefore \quad n = 37$$

To establish the bound $K=2$ used in (2), use the "Closed Interval Method" from Section 4.1 of text:

Let $g(x) = f''(x) = 2e^{-x^2}(2x^2-1)$.

then $g'(x) = 2xe^{-x^2}(6-4x^2) = 0$ at critical numbers $x = 0, \sqrt{\frac{3}{2}}$.

Now test:

	x	$f''(x)$
end point $\to$	0	-2
critical number $\to$	$\sqrt{3/2}$	$4e^{-3/2}$
end point $\to$	2	$14e^{-4}$

$\therefore$ minimum of $f''(x)$ on $[0,2]$ is -2; maximum is $4e^{-3/2}$.

$\therefore$ maximum of $|f''(x)|$ on $[0,2]$ is 2.

A simpler bound on $|f''(x)|$ can be established as follows:

on $[0,2]$, $|f''(x)| = |2e^{-x^2}(2x^2-1)| = 2\underbrace{|e^{-x^2}|}_{\leq e^0}\underbrace{|2x^2-1|}_{\leq 2\cdot 2^2-1} \leq 2 \cdot e^0 \cdot 7 = 14$