Question 1:

(a)[3 points] Evaluate $\int \frac{\tan^{-1} x}{1+x^2}\,dx$.

$u = \tan^{-1} x$

$du = \frac{1}{1+x^2}\,dx$

$\therefore \int \frac{\tan^{-1} x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \boxed{\frac{[\tan^{-1} x]^2}{2} + C}$

(b)[3 points] Suppose $\int_0^5 f'(x)\,dx = 11$, where $f'(x)$ is continuous. If $f(0) = -2$, what is $f(5)$?

$11 = \int_0^5 f'(x)\,dx$

$= f(5) - f(0)$

$= f(5) - (-2)$

$\therefore f(5) = 11 - 2 = \boxed{9}$

(c)[4 points] Suppose the average value of $f(x) = 6x(x-1)$ over the interval $x = 0$ to $x = k$ is k. What is k?

$k = \frac{1}{k-0} \int_0^k 6x(x-1)\,dx$

$= \frac{1}{k} \int_0^k 6x^2 - 6x\,dx$

$= \frac{1}{k} \left[\frac{6x^3}{3} - \frac{6x^2}{2} \right]_0^k$

$= \frac{1}{k} \left[\frac{6k^3}{3} - \frac{6k^2}{2} \right]$

$= 2k^2 - 3k$

$\therefore 2k^2 = 4k$

$\boxed{k = 2}$

Question 2 [10 points]: Evaluate $\frac{9}{4}\int_1^4 \sqrt{t}\ln t\,dt$.

$I = \int t^{1/2}\ln t\,dt \qquad u = \ln t \qquad dv = t^{1/2}dt$

$du = \frac{1}{t}dt \qquad v = \frac{2}{3}t^{3/2}$

$\therefore I = \frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int t^{3/2}\frac{1}{t}\,dt$

$= \frac{2}{3}t^{3/2}\ln t - \frac{2}{3}\int t^{1/2}\,dt$

$= \frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2}$

$\therefore \frac{9}{4}\int_1^4 \sqrt{t}\ln t\,dt = \frac{9}{4}\left[\frac{2}{3}t^{3/2}\ln t - \frac{4}{9}t^{3/2}\right]_1^4$

$= \frac{9}{4}\left[\left(\frac{2}{3}\cdot 4^{3/2}\ln 4 - \frac{4}{9}\cdot 4^{3/2}\right) - \left(\cancel{1\cdot\ln 1}^{\,0} - \frac{4}{9}\cdot 1\right)\right]$

$= \frac{9}{4}\left[\frac{16}{3}\ln 4 - \frac{32}{9} + \frac{4}{9}\right]$

$= \boxed{12\ln 4 - 7}$

Question 3 [10 points]: Evaluate

$$I = \int \frac{x^2}{16\sqrt{16-x^2}}\,dx$$

$$x = 4\sin\theta$$

$$dx = 4\cos\theta\, d\theta$$

$$\therefore I = \int \frac{\cancel{16}\sin^2\theta \; 4\cos\theta\, d\theta}{\cancel{16}\sqrt{16-16\sin^2\theta}}$$

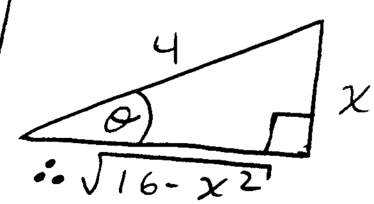
$$= \int \frac{\sin^2\theta \; \cancel{4}\,\cancel{\cos\theta}}{\cancel{4}\,\cancel{\cos\theta}}\, d\theta$$

$$= \int \frac{1}{2} - \frac{\cos(2\theta)}{2}\, d\theta$$

$$= \frac{\theta}{2} - \frac{1}{2}\,\frac{\sin(2\theta)}{2} + C$$

$$= \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta + C$$

$\sin\theta = \frac{x}{4}$:



$$= \frac{\arcsin\left(\frac{x}{4}\right)}{2} - \frac{1}{2}\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + C$$

$$= \boxed{\frac{1}{2}\arcsin\left(\frac{x}{4}\right) - \frac{1}{32}\, x\sqrt{16-x^2} + C}$$

Question 4 [10 points]: Evaluate

$$I = \int \frac{3x^2+8}{x^3+4x}\,dx$$

$$\frac{3x^2+8}{x^3+4x} = \frac{3x^2+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}$$

$$= \frac{(A+B)x^2+Cx+4A}{x(x^2+4)}$$

$\therefore$ $A+B = 3$

$C = 0$

$4A = 8 \longrightarrow \therefore A = 2$; $\therefore$ $B = 3-A = 3-2 = 1$

$$\therefore I = \int \frac{2}{x} + \frac{x}{x^2+4}\,dx$$

$$= 2\int \frac{1}{x}\,dx + \frac{1}{2}\underbrace{\int \frac{2x}{x^2+4}\,dx}_{\substack{u = x^2+4 \\ du = 2x\,dx}}$$

$$= 2\ln|x| + \frac{1}{2}\int \frac{1}{u}\,du$$

$$= 2\ln|x| + \frac{1}{2}\ln|u| + C$$

$$= \boxed{2\ln|x| + \frac{1}{2}\ln|x^2+4| + C}$$

Question 5 [10 points]: Evaluate

$$\int_0^{\pi/3} \sin^7(3t)\cos^3(3t)\,dt$$

$$I = \int \sin^7(3t)\cos^3(3t)\,dt$$

$$= \frac{1}{3}\int \sin^7(3t)\,(1-\sin^2(3t))\,3\cos(3t)\,dt$$

$$u = \sin(3t)$$

$$du = 3\cos(3t)\,dt$$

$$\therefore\ I = \frac{1}{3}\int u^7(1-u^2)\,du$$

$$= \frac{1}{3}\int u^7 - u^9\,du$$

$$= \frac{1}{3}\left[\frac{u^8}{8} - \frac{u^{10}}{10}\right]$$

$$= \frac{1}{3}\left[\frac{\sin^8(3t)}{8} - \frac{\sin^{10}(3t)}{10}\right]$$

$$\therefore\ \int_0^{\frac{\pi}{3}} \sin^7(3t)\cos^3(3t)\,dt$$

$$= \frac{1}{3}\left[\frac{\sin^8(3t)}{8} - \frac{\sin^{10}(3t)}{10}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{3}\left[\left(\cancel{\frac{\sin^8(\pi)}{8}}^{\,0} - \cancel{\frac{\sin^{10}(\pi)}{10}}^{\,0}\right) - \left(\cancel{\frac{\sin^8(0)}{8}}^{\,0} - \cancel{\frac{\sin^{10}(0)}{10}}^{\,0}\right)\right]$$

$$= \boxed{0}$$