Question 1:

(a)[3 points] Let $f(x) = \sinh\left(\cosh^{-1} x\right)$. Find $f'(x)$.

$$f'(x) = \cosh\left(\cosh^{-1} x\right) \cdot \frac{1}{\sqrt{x^2 - 1}}$$

$$= \boxed{\frac{x}{\sqrt{x^2-1}}}$$

(b)[3 points] Find $f'(1)$ if $f(x) = \arctan(\sqrt{x})$.

$$f'(x) = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$f'(1) = \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \boxed{\frac{1}{4}}$$

(c)[4 points] Evaluate $\lim_{x \to \infty} x^2 \sin\left(\frac{1}{x^2}\right)$.

$$\lim_{x \to \infty} x^2 \sin\left(\frac{1}{x^2}\right) = \text{“}\infty \cdot 0\text{”} \quad \Big\} \text{ indeterminate form}$$

$$= \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x^2}\right)}{x^{-2}} \sim \text{“}\frac{0}{0}\text{”} \quad \Big\} \text{ indeterminate form}$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x^2}\right) \cdot \cancel{(-2x^{-3})}}{\cancel{(-2x^{-3})}}$$

$$= \boxed{1}$$

Question 2:

(a)[**3 points**] Find $f(x)$ if $f'(x) = e^{2x} - \frac{1}{\sqrt{x}}$ and $f(0) = 1$.

$$f(x) = \int e^{2x} - x^{-\frac{1}{2}} dx = \frac{e^{2x}}{2} - 2x^{\frac{1}{2}} + C$$

$f(0) = 1$, so $\frac{e^0}{2} - 2 \cdot 0^{\frac{1}{2}} + C = 1 \Rightarrow C = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore f(x) = \frac{e^{2x}}{2} - 2x^{\frac{1}{2}} + \frac{1}{2}$$

(b)[**4 points**] An object initially $s(0) = 2$ m above the surface of the moon is projected vertically upward with an initial velocity of $v(0) = 10$ m/s. Using the fact that acceleration due to gravity on the moon is $a(t) = -1.6$ m/s^2, derive the formula for $s(t)$, the height of the object above the moon's surface at time t seconds.

$a(t) = -1.6$

$\therefore v(t) = \int -1.6\, dt = -1.6t + C$

$v(0) = 10$, so $-1.6 \cdot 0 + C = 10 \Rightarrow C = 10$

$\therefore v(t) = -1.6t + 10$

$\therefore s(t) = \int -1.6t + 10\, dt = \frac{-1.6t^2}{2} + 10t + C$

$s(0) = 2$, so $\frac{-1.6 \cdot 0^2}{2} + 10 \cdot 0 + C = 2 \Rightarrow C = 2$

$$\therefore s(t) = -0.8t^2 + 10t + 2$$

(c)[**3 points**] Suppose $f(x)$ is a continuous function with the property that

$$\int_0^x f(t)\,dt = \sin(2x) - \int_0^x \cos(2t) f(t)\,dt\,.$$

Find a formula for $f(x)$. (Hint: differentiate both sides of the equation above.)

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\left(\sin(2x) - \int_0^x \cos(2t) f(t)\,dt\right)$$

$$f(x) = 2\cos(2x) - \cos(2x) f(x)$$

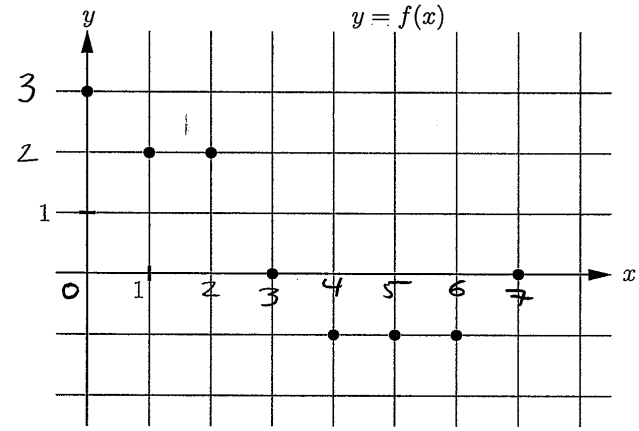
$$\therefore f(x)(1 + \cos(2x)) = 2\cos(2x)$$

$$f(x) = \frac{2\cos(2x)}{1 + \cos(2x)}$$

Question 3:

(a)[**5 points**] The following figure shows points on the graph of $y = f(x)$. Use the trapezoid rule to estimate $\int_0^7 f(x)\,dx$:

$\Delta x = 1$



$$\int_0^7 f(x)\,dx \approx \frac{\Delta x}{2}\left[f(0) + 2f(1) + 2f(2) + \cdots + 2f(6) + f(7)\right]$$

$$= \frac{1}{2}\left[3 + 2\cdot 2 + 2\cdot 2 + 2\cdot 0 + 2\cdot(-1) + 2(-1) + 2(-1) + 0\right]$$

$$= \frac{1}{2}\left[3 + 4 + 4 + 0 + (-2) + (-2) + (-2) + 0\right]$$

$$= \boxed{\frac{5}{2}}$$

(b)[**5 points**] $f(x) = xe^x$ has second derivative $f''(x) = (2 + x)e^x$. If the midpoint rule is being used to approximate $\int_0^2 xe^x\,dx$, how many subintervals are required in order to be accurate to within 0.01? (Recall, the error in using the midpoint rule to approximate $\int_a^b f(x)\,dx$ is at most $\frac{K(b-a)^3}{24n^2}$, where $|f''(x)| \leq K$ on $[a, b]$.)

$|f''(x)| = |(2+x)e^x| \leq (2+2)e^2 = 4e^2$ on $[0,2]$,

so take $K = 4e^2$.

We want $\frac{K(b-a)^3}{24n^2} < 0.01$

$$\frac{4e^2(2-0)^3}{24n^2} < 0.01$$

$$\therefore \frac{4e^2}{(3)(0.01)} < n^2$$

$$\therefore 31.4 < n \qquad \therefore \boxed{n = 32}$$

Question 4:

(a) [**5 points**] Evaluate $\int 3x^2 - x\sin(x^2)\, dx$ $= I$

$\therefore I = 3\int x^2 dx - \frac{1}{2}\int \sin(x^2)\, 2x\, dx$

$= \frac{3x^3}{3} + \frac{1}{2}\cos(x^2) + C$

$= \boxed{x^3 + \frac{1}{2}\cos(x^2) + C}$

(b) [**5 points**] Evaluate $\int_1^e x^{\frac{3}{2}} \ln x\, dx$.

Let $I = \int x^{3/2} \ln x\, dx$

$u = \ln x \qquad dv = x^{3/2} dx$

$du = \frac{1}{x} dx \qquad v = \frac{2}{5} x^{5/2}$

$\therefore I = \int u\, dv = uv - \int v\, du$

$= \frac{2}{5}(\ln x)\, x^{5/2} - \int \frac{2}{5} x^{5/2} \frac{1}{x} dx$

$= \frac{2}{5}(\ln x)\, x^{5/2} - \frac{2}{5}\int x^{3/2} dx$

$= \frac{2}{5}(\ln x)\, x^{5/2} - \left(\frac{2}{5}\right)^2 x^{\frac{5}{2}}$

$\therefore \int_1^e x^{\frac{3}{2}} \ln x\, dx = \left[\frac{2}{5}(\ln x)\, x^{5/2} - \left(\frac{2}{5}\right)^2 x^{5/2}\right]_1^e$

$= \left[\frac{2}{5} e^{5/2} - \left(\frac{2}{5}\right)^2 e^{5/2}\right] - \left[0 - \left(\frac{2}{5}\right)^2\right]$

$= \boxed{\frac{6}{25} e^{\frac{5}{2}} + \frac{4}{25}}$

Question 5:

(a) [**5 points**] Evaluate $\int \tan^2 x \sec^4 x \, dx = I$

$$I = \int \tan^2 x \, (1+\tan^2 x) \sec^2 x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$\therefore I = \int u^2 (1+u^2) \, du$$

$$= \int u^2 + u^4 \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$$

(b) [**5 points**] Evaluate $\int \frac{\sqrt{x^2-1}}{x^3} \, dx$. (The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ may be useful here.) $= I$

$x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$$\therefore I = \int \frac{\sqrt{\sec^2\theta - 1} \, \sec\theta \tan\theta}{\sec^3\theta} \, d\theta$$

$$= \int \frac{\tan^2\theta \sec\theta}{\sec^3\theta} \, d\theta$$

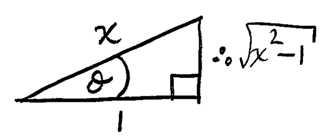
$$= \int \sin^2\theta \, d\theta$$

$$= \int \frac{1}{2} - \frac{\cos(2\theta)}{2} \, d\theta$$

$$= \frac{\theta}{2} - \frac{1}{2}\frac{\sin(2\theta)}{2} + C$$

$$= \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta + C$$

$\sec\theta = \frac{x}{1}$, so



$$\therefore I = \frac{\sec^{-1}(x)}{2} - \frac{1}{2}\frac{\sqrt{x^2-1}}{x}\frac{1}{x} + C$$

$$= \boxed{\frac{\sec^{-1}(x)}{2} - \frac{\sqrt{x^2-1}}{2x^2} + C}$$

Question 6:

(a)[5 points] Evaluate $\int \frac{1}{x^3+4x^2}\,dx = I$

$$\frac{1}{x^3+4x^2} = \frac{1}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$= \frac{Ax(x+4) + B(x+4) + Cx^2}{x^2(x+4)}$$

$$= \frac{(A+C)x^2 + (4A+B)x + 4B}{x^2(x+4)}$$

$\therefore$ ① $A+C=0$

② $4A+B=0$

③ $4B=1 \Rightarrow B=\frac{1}{4}$; ② $\Rightarrow 4A+\frac{1}{4}=0$, $\therefore A=\frac{-1}{16}$

$\therefore$ ① $\Rightarrow C=-A=\frac{1}{16}$.

$$\therefore I = \int \frac{(-1/16)}{x} + \frac{(1/4)}{x^2} + \frac{(1/16)}{x+4}\,dx$$

$$= \boxed{-\frac{1}{16}\ln|x| - \frac{1}{4}\frac{1}{x} + \frac{1}{16}\ln|x+4| + C}$$

(b)[5 points] Evaluate the improper integral $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

$$\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx = \lim_{b\to 3^-}\int_0^b \frac{x}{\sqrt{9-x^2}}\,dx$$

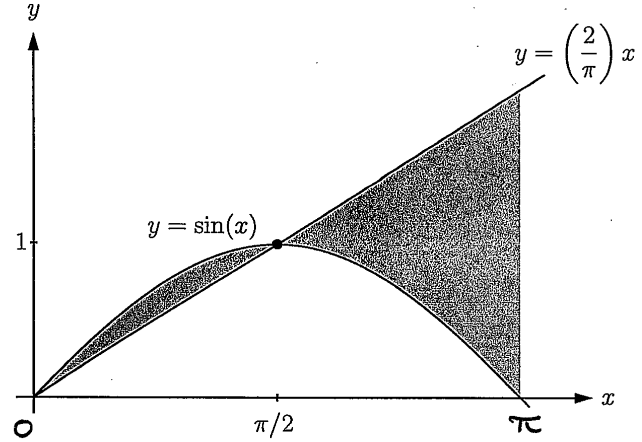
$$= \lim_{b\to 3^-}\left[-\sqrt{9-x^2}\right]_0^b$$

$$= \lim_{b\to 3^-}\left(-\sqrt{9-b^2} + \sqrt{9-0^2}\right)$$

$$= \boxed{3}$$

Question 7:

(a) [**5 points**] Find the area of the shaded region:



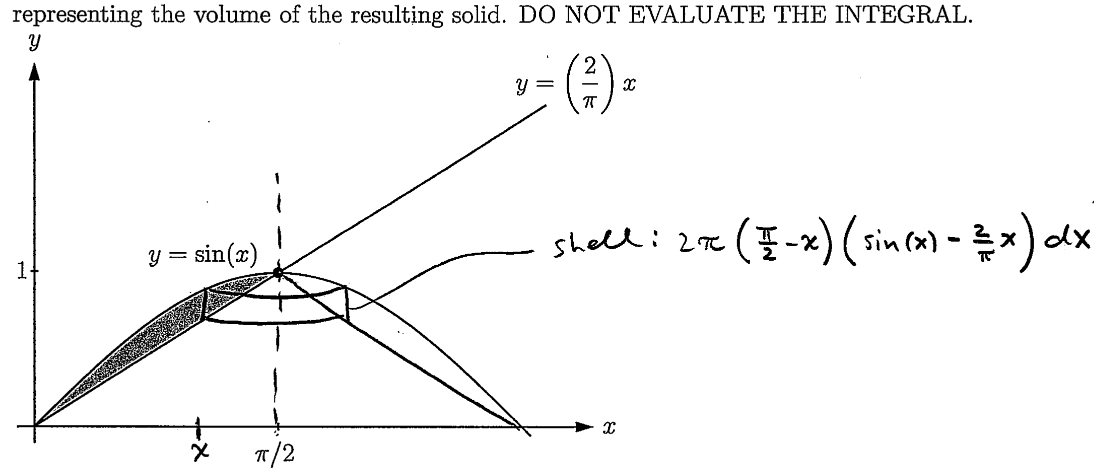
$$A = \int_0^{\frac{\pi}{2}} \sin x - \frac{2}{\pi}x \, dx + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{2}{\pi}\right)x - \sin x \, dx$$

$$= \left[-\cos x - \frac{1}{\pi}x^2\right]_0^{\frac{\pi}{2}} + \left[\frac{1}{\pi}x^2 + \cos x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left[\left(-\cos\left(\frac{\pi}{2}\right) - \frac{1}{\pi}\left(\frac{\pi}{2}\right)^2\right) - \left(-\cos(0) - 0\right)\right] + \left[\left(\frac{1}{\pi}\pi^2 + \cos(\pi)\right) - \left(\frac{1}{\pi}\left(\frac{\pi}{2}\right)^2 + \cos\frac{\pi}{2}\right)\right]$$

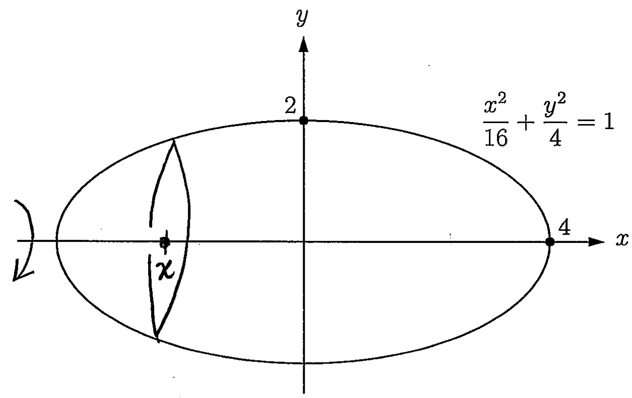
$$= -\frac{\pi}{4} + 1 + \pi - 1 - \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$

(b) [**5 points**] The shaded region is rotated about the vertical line $x = \pi/2$; set up the integral representing the volume of the resulting solid. DO NOT EVALUATE THE INTEGRAL.



$$\therefore V = \int_0^{\frac{\pi}{2}} 2\pi\left(\frac{\pi}{2}-x\right)\left(\sin(x) - \frac{2}{\pi}x\right)dx$$

Question 8: Consider the graph of the following ellipse:



If the ellipse is rotated about the x-axis the resulting solid is called an ellipsoid (which looks rather like a watermellon).

(a) [**3 points**] Isolate y in the equation above to find a function which describes the top half of the ellipse.

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\therefore \quad y = \sqrt{4\left(1 - \frac{x^2}{16}\right)}$$

(b) [**7 points**] Use your result in (a) to find the volume of the ellipsoid.

Using disks:

$$V = \int_{-4}^{4} \pi \left(\sqrt{4\left(1 - \frac{x^2}{16}\right)}\right)^2 dx$$

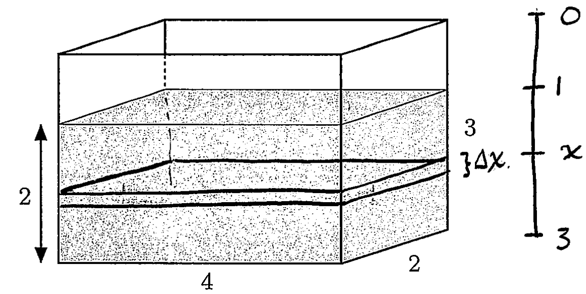
$$= 2\pi \int_0^4 4\left(1 - \frac{x^2}{16}\right) dx$$

$$= 8\pi \left[x - \frac{x^3}{48}\right]_0^4$$

$$= 8\pi \left[4 - \frac{64}{48}\right]$$

$$= \boxed{\frac{64\pi}{3}}$$

Question 9: A rectangular fish tank of length 4 m, width 2 m and height 3 m contains water to a depth of 2 m. Recall that the density of water is $\rho = 1000$ kg/m^3 and acceleration due to gravity is $g = 9.8$ m/s^2.



(a) [**5 points**] How much work is required to pump all of the water out over the edge of the tank?

Weight of slice of water a depth x = $(4)(2)\Delta x \rho g$

$\therefore$ Work required to lift this slice to top of tank is $(4)(2)\Delta x \rho g\, x = 8\rho g x \,\Delta x$

$\therefore$ Total work $W = \int_{x=1}^{x=3} 8\rho g x \, dx$

$= 8\rho g \left[\frac{x^2}{2}\right]_1^3$

$= 8\rho g \,\frac{9-1}{2}$

$= 32\rho g$

$= (32)(1000)(9.8)$

$= \boxed{313{,}600 \text{ N}\cdot\text{m}}$

(b) [**5 points**] What is the hydrostatic force (force due to water pressure) exerted on one of the long sides of the tank? Recall that pressure P as a function of depth h is $P(h) = \rho g h$ where ρ is the density of the liquid and g is acceleration due to gravity.

area of strip of width Δx is $4\Delta x$

$\therefore$ force on this strip is $4\Delta x \rho g (x-1)$

$\therefore$ Total force $F = \int_{x=1}^{x=3} 4\rho g (x-1)\, dx$

$= 4\rho g \int_1^3 x-1 \, dx$

$= 4\rho g \left[\frac{x^2}{2} - x\right]_1^3$

$= 4\rho g \left[\frac{3}{2} + \frac{1}{2}\right]$

$= 8\rho g = (8)(1000)(9.8) = \boxed{78400 \text{ N}}$

Question 10: The fish population in a large lake is infected by a disease at time $t = 0$, and the declining fish population is described by the differential equation

$$\frac{dP}{dt} = -k\sqrt{P}\,.$$

Here k is a positive constant and $P(t)$ is the fish population at time t weeks. Suppose there were initially 90,000 fish in the lake and that 40,000 remain after 6 weeks.

(a) [**7 points**] Solve the differential equation to find a formula for $P(t)$.

$$\int \frac{1}{\sqrt{P}}\,dP = \int -k\,dt$$

$$2\sqrt{P} = -kt + C$$

$P(0) = 90{,}000$, so $2\sqrt{90{,}000} = -k\cdot 0 + C$

$\therefore\ C = 600$

$$\therefore 2\sqrt{P} = -kt + 600$$

$P(6) = 40{,}000$, so $2\sqrt{40{,}000} = -k\cdot 6 + 600$

$\therefore\ k = \dfrac{600 - 2\cdot 200}{6} = \dfrac{100}{3}$

$$\therefore\ 2\sqrt{P} = 600 - \frac{100}{3}t$$

$$\boxed{P(t) = \left(300 - \frac{50}{3}t\right)^2}$$

(b) [**3 points**] Use your result in (a) to find the time required for the fish population to reduce to 10,000.

Solve $\left(300 - \frac{50}{3}t\right)^2 = 10{,}000$

$$300 - \frac{50}{3}t = 100$$

$$200 = \frac{50}{3}t$$

$$t = \frac{3\cdot 200}{50} = \boxed{12 \text{ weeks}}$$

Question 11: Recall that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(a) [**4 points**] Find the first three non-zero terms of the Maclaurin series for $f(x) = \sinh(x^2)$.

$$\sinh(x^2) = \frac{e^{x^2} - e^{-x^2}}{2}$$

$$= \frac{\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) - \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right)}{2}$$

$$= \boxed{x^2 + \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots}$$

(b) [**3 points**] Use a Maclaurin series to evaluate the limit

$$\lim_{x \to 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots\right) - 1 - x - \frac{x^2}{2}}{x^3}$$

$$= \lim_{x \to 0} \frac{1}{3!} + \frac{x^2}{4!} + \cdots$$

$$= \boxed{\frac{1}{6}}$$

(c) [**3 points**] Suppose $f(x)$ is a function such that $f(2) = 3$, $f'(2) = 0$, $f''(2) = -1$ and $f'''(2) = 2$. Use a Taylor polynomial of degree 3 to approximate $f(2.1)$. Round your final answer to three decimals.

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= 3 - \frac{1}{2}(x-2)^2 + \frac{2}{3!}(x-2)^3$$

$$\therefore f(2.1) \approx T_3(2.1) = 3 - \frac{1}{2}(0.1)^2 + \frac{2}{3!}(0.1)^3$$

$$= \boxed{2.995}$$