The following problems are good practice for the final exam. Some of these problems are challenging but completely within the scope of the material we have covered. The problems on the final exam may of course be completely different.

Shorter Problems

1. Evaluate
$$\int \frac{\ln x}{\sqrt{x}}\,dx$$

2. Find a formula for y as a function of x if
$$\frac{dy}{dx} = xe^{-y}, \qquad y(2) = 0\ .$$

3. A mine is producing silver at an instantaneous rate of
$$f(t) = \frac{500{,}000}{1+t^2}$$
ounces per year for $t \geq 0$, where t is time in years and $t = 0$ represents the present. Assume that the market price of silver at time t is $g(t) = 10e^{0.03t}$ dollars per ounce. Write down an integral which gives the future value at $t = 30$ of all the silver the mine will produce during the next 30 years, assuming that a safe investment is available which pays 7% interest compounded continuously. *Do not attempt to evaluate this integral.*

4. Suppose the matrix $\begin{bmatrix} 2 & -3 \\ k & 3 \end{bmatrix}$ is not invertible. What is k?

5. The interest rate i expressed as a percentage is a function of time t. Assume that $di/dt = 1$ for $1 \leq t < 2$ and $di/dt = t - 1$ for $2 \leq t \leq 3$. Find the change in the interest rate between $t = 1$ and $t = 3$.

6. Evaluate
$$\int_1^\infty \frac{\ln x}{x^3}\,dx$$
(For this problem recall that $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any $p > 0$.)

7. Evaluate
$$\int \frac{x+2}{(x-1)^{3/2}}\,dx$$

8. Suppose the instantaneous rate in kilograms per month at which an animal is gaining weight is denoted by a function $f(t)$. Assume that $f(1) = 2$, $f(1.5) = 0$, $f(2.5) = 3$ and $f(4.5) = 5$. Use trapezoids to estimate the total weight gained by the animal from $t = 1$ to $t = 4.5$.

9. Evaluate
$$\int_1^2 \frac{2+x}{x+x^2}\,dx\ .$$

10. Find the critical points of $f(x, y) = y^3 - 3xy + 6x$.

11. The following facts are known about the function $f(t)$: (i) $f''(t) = 16e^{-2t}$ for all t; (ii) $f'(0) = 2$; (iii) $f(0) = 3$. Find a formula for $f(t)$.

12. When the price of a commodity is p, the quantity demanded is q, where p and q are related by the demand equation

$$p = \frac{15}{(3q + 1)^{1/2}}$$

Find the consumer surplus when the price p is equal to 3. Please simplify.

13. Suppose $\mathbf{A}$ and $\mathbf{B}$ are column matrices with 5 entries each. What is the size of $\mathbf{AI}(\mathbf{A}+\mathbf{B})^{\mathrm{T}}\mathbf{IB}$?

14. Compute $\displaystyle\int_1^e \frac{\ln x}{x^2}\, dx$

15. Compute $\displaystyle\int \frac{x}{(1 + x)(3 - 2x)}\, dx$

16. Let $f(x) = \dfrac{4}{5 - 2x}$. Find the average value of $f(x)$ over the interval $0 \le x \le 2$.

17. The demand q for a product is related to the unit price p by the demand equation $p = D(q) = 10(1 + 4q)^{-1/2}$. Find the consumer surplus if $p = 2$. Please simplify. Note that if $p = 2$ then $q = 6$.

18. Find a formula for y as a function of x if

$$\frac{dy}{dx} = -2xy^2e^{x^2}, \qquad y(0) = -1/2\ .$$

19. Compute $f_x(1, 0) + f_y(1, 0) - f_{xy}(1, 0)$ if $f(x, y) = 10x^2e^{3y}$.

20. Suppose $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices. Simplify

$$\left(\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}\right)(2\mathbf{A} - 2\mathbf{B}) - \mathbf{BA}$$

Longer Problems

1. Let $w(t)$ be the weight of an animal at time t measured in years, where $t = 0$ represents the present. Suppose it is known that $w(0) = 5$ and that

$$\frac{dw}{dt} = \frac{100t}{(t^2 + 1)^3}$$

for $t \ge 0$. When will the animal weigh 29 kilograms?

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

(a) Find $\mathbf{A^{-1}}$.

(b) Use the inverse in part (a) to solve the system of equations $\mathbf{AX} = \mathbf{Y}$ where $\mathbf{X}^{\mathrm{T}} = [\ x_{11}\ \ x_{21}\ \ x_{31}\ ]$ and $y_{11} = 1$, $y_{21} = 4$, $y_{31} = 1$.

3. The enrollment $N = N(t)$ at a college is predicted to change at a rate of

$$N'(t) = \frac{2000}{(1 + 0.2t)^{3/2}} \ ,$$

where t is time measured in years and $t = 0$ represents the present. When will the enrollment reach 11,000 if the current enrollment is 1000?

4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & -2 \\ 2 & 3 & -5 \end{bmatrix}$$

Suppose also that $\mathbf{B}$ is the matrix of size 3×1 whose entries are $b_{11} = -2$, $b_{21} = 1$ and $b_{31} = -3$.

(a) Find the inverse of $\mathbf{A}$ or show that it does not exist.

(b) Find all solutions to $\mathbf{AX} = \mathbf{B}$ where $\mathbf{X}^{\mathrm{T}} = [\ x\ \ y\ \ z\ ]$.

5. An oil tanker entering a harbour is leaking oil in barrels per hour at a rate given by

$$A'(t) = \frac{80 \ln(t + 1)}{t + 1} \ ,$$

where $t = 0$ represents the time the tanker hits a hidden rock. How many barrels of oil does the tanker leak between $t = 5$ and $t = 10$ hours?

6. A business has just opened and it is estimated that it will produce income at an instantaneous rate of $f(t) = 72t$ dollars per year, where t is the time in years and $t = 0$ corresponds to the present. What is the present value of all the income that the business will ever produce? Assume that a safe investment is available which pays 6% compounded continuously.

7. Let k be a constant and consider the system of linear equations

$$\begin{aligned} 2x - y + 3z &= -2 \\ x + y + 2z &= 4 \\ 3x - 3y + 4z &= k \end{aligned}$$

(a) For what value of k does the system of equations have solutions?

(b) For the value of k in part (a) find the solution corresponding to $z = 1$.

8. Use the Trapezoidal Rule to estimate the area of the region that lies below the curve $y = f(x)$, above $y = 2 - 2x$, and between $x = 1$ and $x = 3$. All of the known information about $f(x)$ is given in the table below.

x	1	1.5	2.0	2.5	3.0
$f(x)$	2	3	5	4	2

9. Let $f(t) = 1000$ for $0 \leq t \leq 2$ and $g(t) = 200$ for $t \geq 0$ be two continuous income streams, where t is time measured in years. At what nominal interest rate under continuous compounding would these two income streams be of equal value? Report your answer as a percentage to 3 significant figures.

10. The total surplus is defined to be the sum of the consumers' surplus and the producers' surplus. Find the total surplus if the demand function is $p = 64 - q^2$ and the supply function is $p = q^2 + 2q + 24$, where, as usual, q is the quantity and p is the corresponding unit price in dollars. Report your answer to the nearest cent.

11. The temperature T at a weather station is well approximated by the formula

$$T(t) = \frac{4t}{\sqrt{2t+1}}$$

for $0 \leq t \leq 3/2$, where t is time measured in hours. Find the average temperature over the one and a half hour period.

12. The profit, in dollars, that a business generates is a continuous income stream with flow rate $f(t) = Ct$ for $t \geq 0$. Here t is time in years, $t = 0$ represents the present, and C is a constant. If the nominal rate of interest is 7% with continuous compounding, determine the value of C that makes \$5,000,000 the current fair market price for the purchase of this business.

13. Supply and demand determine the equilibrium price p_e for a certain commodity. Assume that $p_e =$ \$50 but the present market price is \$60. Let time be measured in months, and suppose that the market price p is decreasing toward the equilibrium price at a rate which is proportional to the difference between the market price and the equilibrium price with proportionality constant 1/10. How long will it take, to the nearest tenth of a month, until the market price reaches \$55?

14. If a certain company sells x units of product X and y units of product Y, its profit $P(x, y)$ is given by the formula

$$P(x, y) = 120x + 90y - 4x^2 - xy - y^2 \ .$$

How many units of X and how many units of Y should the company sell to maximize profit? Omit the justification that your answers actually yield the maximum profit.

15. Let $R(q)$ be the revenue, in millions of dollars, that a refinery obtains from selling q million barrels of fuel oil. Let $C(q)$ be the cost, in millions of dollars, of producing q million barrels. The table below gives the estimated marginal revenue dR/dq and marginal cost dC/dq at various levels q of production.

q	0	5	10	15	20
Marginal Revenue	40	40	40	40	38
Marginal Cost	36	36	36	37	38

The fixed costs are 10 (million dollars). Use the Trapezoidal Rule to estimate the profit made by the refinery from producing and selling 20 million barrels of fuel oil. Find an explicit numerical answer. Be sure to write down the integral or integrals that are being approximated by the Trapezoidal Rule.

16. By using x units of capital and y units of labour, a manufacturer can produce $Q(x, y)$ kilograms of a pesticide, where

$$Q(x, y) = \frac{xy}{3x + 2y}$$

A unit of capital costs 1 dollar, and a unit of labour costs 6 dollars. Use the method of Lagrange Multipliers to determine the maximum amount of the pesticide that can be produced for a total expenditure of \$800. Credit will only be given for a method that uses Lagrange Multipliers. Omit the justification that your answer actually is the maximum amount.

17. At any time t, where $t \geq 0$, water is flowing into a reservoir at the instantaneous rate $W'(t)$, where

$$W'(t) = \frac{t}{(t^2 + 4)^2}$$

In the above formula, t is measured in months, and W in millions of cubic meters.

(a) How much water flows into the reservoir from time $t = 0$ to time $t = 1$?

(b) Suppose that at time $t = 0$ the reservoir contains 0.5 million cubic meters of water. What is the average amount of water in the reservoir over the time interval $0 \leq t \leq 2$? Please simplify as much as possible.

18. There was an outbreak of salmon influenza at a fish farm. At the instant the epidemic began, there were 400 kilograms of live salmon fingerling at the farm. Five days later there were only 100 kilograms of live fingerling left. Let $W = W(t)$ be the amount, in kilograms, of live fingerling t days after the outbreak of the disease. Assume that

$$\frac{dW}{dt} = -kW$$

for some positive constant k. Use the given information to evaluate k, and find an explicit formula for W as a function of t.

19. Solve the differential equation

$$\frac{dK}{dt} = 60 - 0.05K, \qquad K(0) = 1000 .$$

20. Let A be the part of the first quadrant that lies on or below the line $y = 3$ and on or above the curve $y = \sqrt{1 + x^3}$. Use the trapezoidal rule with $n = 2$ to estimate the area of A.