3 Matrix Multiplication

3.1 Introduction

So far we have seen two algebraic operations with matrices, addition and scalar multiplication, and we have also seen how the zero matrix plays a role in these operations similar to that of the zero of the real number system. The properties of matrix addition and scalar multiplication are similar to those of the ordinary real numbers, and it is natural to ask how far these similarities extend. In particular, is it possible the define the notion of multiplication of two matrices. The answer is yes, though matrix multiplication is not quite as straightforward as matrix addition.

3.2 Definition of Matrix Multiplication

Here's the formal definition of matrix multiplication:

Definition: Let $\mathbf{A} = [a_{ij}]_{m\times p}$ and $\mathbf{B} = [b_{ij}]_{p\times n}$. Then we define the product $\mathbf{C} = \mathbf{AB}$ as the matrix with entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj} \ .$$

Notice:

(i) $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{ip}$ are the elements of row i of $\mathbf{A}$, while $b_{1j}, b_{2j}, b_{3j}, \ldots, b_{pj}$ are the elements of column j of $\mathbf{B}$.

(ii) For matrix multiplication to be defined, the number of columns of $\mathbf{A}$ must be the same as the number of rows of $\mathbf{B}$.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 6 & 4 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 3 & 7 \\ -2 & 5 \end{bmatrix}.$$

Then

$$\mathbf{A}\,\mathbf{B} = \begin{bmatrix} (2)(3)+(3)(-2) & (2)(7)+(3)(5) \\ (1)(3)+(-2)(-2) & (1)(7)+(-2)(5) \\ (6)(3)+(4)(-2) & (6)(7)+(4)(5) \end{bmatrix} = \begin{bmatrix} 0 & 29 \\ -1 & -3 \\ 10 & 62 \end{bmatrix}.$$

Notice $\mathbf{A}$ is size 3×2, $\mathbf{B}$ is size 2×2, and the product $\mathbf{AB}$ is size 2×2. □

The process of multiplying matrices can be described in terms of a certain type of product. Let **R** and **C** be the row and column matrices

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{p1} \end{bmatrix} .$$

The expression $r_{11}c_{11} + r_{12}c_{21} + \cdots + r_{1p}c_{p1}$ is called the ***dot product*** of **R** and **C**. Using this notion, the product of general matrices **A** and **B** can be described as the matrix $\mathbf{C} = \mathbf{AB}$ which has entries

$$c_{ij} = \text{ dot product of row } i \text{ of } \mathbf{A} \text{ and column } j \text{ of } \mathbf{B} .$$

3.3 Examples of Matrix Multiplication

Here are some examples to illustrate the properties of matrix multiplication:

Example:

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} -2 & 13 \end{bmatrix} = \begin{bmatrix} -6 & 39 \\ -10 & 65 \end{bmatrix} .$$

□

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} .$$

Then

$$\mathbf{AB} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 13 & 13 \end{bmatrix} .$$

Now switch the order of the two factors:

$$\mathbf{BA} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 32 \\ 3 & 16 \end{bmatrix} .$$

□

Notice in this last example: $\mathbf{AB} \neq \mathbf{BA}$. That is, matrix multiplication is not commutative. This is a very important difference between matrix multiplication and the ordinary multiplication of real numbers that we are used to.

Example: Again let

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} .$$

Then

$$(\mathbf{AB})^{\mathrm{T}} = \left(\begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \right)^{\mathrm{T}} = \begin{bmatrix} 9 & 9 \\ 13 & 13 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 9 & 13 \\ 9 & 13 \end{bmatrix} .$$

Now compute

$$\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 9 & 13 \end{bmatrix} .$$

□

In this last example we see that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, and this turns out to be true in general.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \qquad \mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Then

$$\mathbf{AI_2} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} = \mathbf{A} .$$

Similarly,

$$\mathbf{I_2A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 2 & 9 \end{bmatrix} = \mathbf{A} .$$

□

The matrix $\mathbf{I_2}$ in this last example is called the ***identity matrix*** of order 2. More generally:

Definition: The $n \times n$ matrix

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is called the ***identity matrix of order*** n, or the $n \times n$ ***identity matrix***. It has the property that

$$[a_{ij}]_{m\times p}\mathbf{I_p} = [a_{ij}]_{m\times p} ,$$

and

$$\mathbf{I_m}[a_{ij}]_{m\times p} = [a_{ij}]_{m\times p} .$$

When the size of the identity matrix is understood it is often written simply as $\mathbf{I}$. The letter $\mathbf{I}$ should not be used to represent any other matrix— it always means the identity matrix.

Comparing matrix algebra with that of the ordinary real numbers, notice that $\mathbf{I_n}$ plays the role of the number one.

As with matrix addition, matrix multiplication has many of the properties of ordinary multiplication of real numbers. Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be matrices, $\mathbf{I}$ the identity, and k a scalar. Then

(i) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

(ii) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

(iii) $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$.

(iv) $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = (\mathbf{A}k)\mathbf{B} = \mathbf{A}(k\mathbf{B}) = \mathbf{A}(\mathbf{B}k)$.

(v) $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

(vi) $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

(vii) If $p \geq 1$ is an integer, and $\mathbf{A}$ is square, $\mathbf{A}^p = \underbrace{\mathbf{AA}\cdots\mathbf{A}}_{p \text{ factors}}$.

(viii) $\mathbf{A}^0 = \mathbf{I}$.

In the properties listed above it is assumed that the matrices are of the sizes required for the operations to be defined. Notice how the order of the multiplication is preserved in the expansion in *(ii)* and *(iii)*. This is important since matrix multiplication is not commutative in general.

Example: Expand $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$.

$$\begin{aligned}(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) &= (\mathbf{A}+\mathbf{B})\mathbf{C} + (\mathbf{A}+\mathbf{B})\mathbf{D} \\ &= \mathbf{AC}+\mathbf{BC}+\mathbf{AD}+\mathbf{BD} .\end{aligned}$$

□

Example: Expand $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$.

$$\begin{aligned}(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) &= (\mathbf{A}+\mathbf{B})\mathbf{A} - (\mathbf{A}+\mathbf{B})\mathbf{B} \\ &= \mathbf{A}^2+\mathbf{BA}-\mathbf{AB}-\mathbf{B}^2 .\end{aligned}$$

□

Notice in this last example, the terms $\mathbf{BA}$ and $-\mathbf{AB}$ cannot be canceled unless $\mathbf{AB} = \mathbf{BA}$, which is not true in general.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 13 & 5 \\ 1 & 8 & 3 \\ 1 & 3 & 1 \end{bmatrix} .$$

Then

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 13 & 5 \\ 1 & 8 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

That is, $\mathbf{AB} = \mathbf{I}$. □

In this last example, $\mathbf{B}$ is called the ***inverse*** of $\mathbf{A}$, and it is the only matrix with this property. You will find that $\mathbf{BA} = \mathbf{I}$ in this case also. The inverse of $\mathbf{A}$ is written $\mathbf{A^{-1}}$. We will explore inverses in more detail later.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}.$$

Then

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

In this example we see another aspect of matrix multiplication which is unlike our experience with ordinary multiplication: it is possible that $\mathbf{AB} = \mathbf{0_{2\times 2}}$ even though $\mathbf{A} \neq \mathbf{0_{2\times 2}}$ and $\mathbf{B} \neq \mathbf{0_{2\times 2}}$.

Problems for Section 3

1. Let

$$\mathbf{A} = \begin{bmatrix} -3 & -7 & -6 \\ -2 & 4 & 6 \\ 5 & 4 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -2 & 5 \\ 4 & 2 & -7 \\ -4 & -4 & -4 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 2 & 4 & 4 \\ 2 & -6 & 3 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 7 & -7 \\ -7 & -6 \\ -3 & -3 \end{bmatrix}$$

   Compute the following:

   (i) $\mathbf{AB}$

   (ii) $\mathbf{BD}$

   (iii) $3\mathbf{I} + \frac{3}{2}\mathbf{CD}$

   (iv) $\mathbf{DC}(\mathbf{B} - \mathbf{A})$

   (v) $(\mathbf{A} - 2\mathbf{I})(\mathbf{B} + 3\mathbf{I})$

   (vi) $(\mathbf{CA})^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}$

2. Express the system of equations

$$5x - 2y + z = -2, \qquad -x + 11y - 13z = 6, \qquad x + y + z = 1$$

   using matrix multiplication.

3. Express the system of equations

$$-x+3z=5, \qquad 7x=6y+3, \qquad y-z=0$$

   using matrix multiplication.

4. For the diagonal matrices

$$\mathbf{A}=\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix},$$

   compute **AB** and also **BA**. Notice anything?

5. Let

$$\mathbf{A}=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$$

   Compute $(\mathbf{AB})^2$.

Solutions to Problems for Section 3

1. *(i)*

$$\mathbf{AB}=\begin{bmatrix} -4 & 16 & 58 \\ -8 & -12 & -62 \\ 36 & 18 & 17 \end{bmatrix}$$

   (ii)

$$\mathbf{BD}=\begin{bmatrix} -1 & -3 \\ 35 & -19 \\ 12 & 64 \end{bmatrix}$$

   (iii)

$$3\mathbf{I}+\frac{3}{2}\mathbf{CD}=\begin{bmatrix} -36 & -75 \\ \frac{141}{2} & \frac{45}{2} \end{bmatrix}$$

   (iv)

$$\mathbf{DC}(\mathbf{B}-\mathbf{A})=\begin{bmatrix} 357 & -196 & -903 \\ 384 & 222 & -436 \\ 189 & 96 & -231 \end{bmatrix}$$

(v)

$$(\mathbf{A} - 2\mathbf{I})(\mathbf{B} + 3\mathbf{I}) = \begin{bmatrix} -19 & -1 & 30 \\ -22 & -10 & -30 \\ 59 & 38 & 4 \end{bmatrix}$$

(vi)

$$(\mathbf{CA})^{\mathrm{T}} - \mathbf{A}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{0_{3\times 2}}$$

2.

$$\begin{bmatrix} 5 & -2 & 1 \\ -1 & 11 & -13 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$$

3.

$$\begin{bmatrix} -1 & 0 & 3 \\ 7 & -6 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

4.

$$\mathbf{AB} = \mathbf{BA} = \begin{bmatrix} ax & 0 & 0 \\ 0 & by & 0 \\ 0 & 0 & cz \end{bmatrix}$$

5.

$$(\mathbf{AB})^2 = \begin{bmatrix} 48 & -32 & 16 \\ -24 & 16 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$