1 Matrices: Introduction, Terminology, Notation

1.1 Introduction

Consider the problem of solving the system of equations

$$x - 2y = 1 \tag{1}$$
$$-2x + 3y = -2 \tag{2}$$

We can do this easily using substitution: use equation (1) to write $x = 1 + 2y$, and then substitute this expression into equation (2) to get

$$-2(1 + 2y) + 3y = -2\ ,$$

which is easily solved to find $y = 0$. Now substitute $y = 0$ into the equation $x = 1 + 2y$ to find $x = 1$. Therefore, the solution to the system of equations is $x = 1$ and $y = 0$.

This works fine for this small system of two equations. What about something larger, say

$$\left.\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 1 \\ -2x & + & 3y & + & z & = & -2 \\ 5x & - & 7y & - & 3z & = & -3 \end{array}\right\} \tag{3}$$

We could again use substitution, but the problem is quite a bit harder (and messier) this time—try it! The situation gets worse the more equations we have. We would like a more systematic and organized approach to solving such problems, and fortunately, there is one. The system (3) can be expressed in the form

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} .$$

Each of the rectangular structures $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix}$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$ is called a ***matrix***, and there are methods of manipulating them to efficiently solve the system of equations. Our main focus will be the application of matrices (the plural of matrix) to solving systems of equations, but it should noted that matrices arise in many areas of science, economics and computing.

1.2 Terminology and Notation

Matrices are denoted using bold uppercase letters. For example, let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} .$$

Here $\mathbf{A}$ is a matrix of 3 ***rows*** and 3 ***columns***, and we say $\mathbf{A}$ has ***size*** 3×3. The size of a matrix is always stated as *rows*×*columns*, that is, rows first, columns second.

The ***entries*** or ***elements*** of $\mathbf{A}$ are the numbers. These are denoted by their position:

$$a_{ij} = \text{ entry in row } i \text{ and column } j \text{ of } \mathbf{A}\ .$$

For example, in our matrix $\mathbf{A}$, $a_{31} = 5$ since the entry in row 3 and column 1 is 5. Notice again, when referring to position of entries: rows first, columns second. Similarly, $a_{23} = 1$ is the entry in row 2 column 3. Also notice how we have used the lowercase letter "a" in a_{ij} to correspond with our choice of uppercase "$\mathbf{A}$" used to represent the matrix. If instead we used $\mathbf{B}$ to represent our matrix, we would use b_{ij} to refer to the individual entries.

Now that we have some notation, let's state the formal definition of a matrix:

Definition: A rectangular array of numbers consisting of m horizontal rows and n vertical columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an $m \times n$ matrix or matrix of size $m \times n$. For entry a_{ij}, i is the row subscript, while j is the column subscript.

A general matrix is sometimes denoted $[a_{ij}]_{m \times n}$.

Example: Let

$$\mathbf{B} = \begin{bmatrix} 1 & 6/7 & 4 \\ 1/2 & -7 & 3 \end{bmatrix} .$$

The size of $\mathbf{B}$ is 2×3. A couple of entries of $\mathbf{B}$ are

$$b_{23} = 3 \qquad b_{21} = 1/2\ .$$

□

Example: Let

$$\mathbf{P} = \begin{bmatrix} 3 & -4 & \pi \end{bmatrix} .$$

$\mathbf{P}$ is called a ***row matrix*** or ***row vector***. Here $p_{11} = 3$, $p_{12} = -4$ and $p_{13} = \pi$.

□

Example: Let

$$\mathbf{Q} = \begin{bmatrix} 0 \\ 2 \\ e \end{bmatrix} .$$

$\mathbf{Q}$ is called a ***column matrix*** or ***column vector***. Here $q_{11} = 0$, $p_{21} = 2$ and $p_{31} = e$.

□

Example: Construct $[a_{ij}]_{4\times 3}$ if $a_{ij} = \dfrac{1}{i+j}$.

Solution: Let $\mathbf{A} = [a_{ij}]_{4\times 3}$. The entries of $\mathbf{A}$ are functions of their row and column positions; that is, $a_{ij} = 1/(i+j)$ is a function of the two variables $i = 1, 2, 3, 4$ and $j = 1, 2, 3$:

$$a_{11} = \frac{1}{1+1} = \frac{1}{2}\,,$$

$$a_{12} = \frac{1}{1+2} = \frac{1}{3}\,,$$

$$a_{13} = \frac{1}{1+3} = \frac{1}{4}\,,$$

and so on. Computing all of the entries in this way we have

$$\mathbf{A} = \begin{bmatrix} \frac{1}{1+1} & \frac{1}{1+2} & \frac{1}{1+3} \\ \frac{1}{2+1} & \frac{1}{2+2} & \frac{1}{2+3} \\ \frac{1}{3+1} & \frac{1}{3+2} & \frac{1}{3+3} \\ \frac{1}{4+1} & \frac{1}{4+2} & \frac{1}{4+3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}.$$

□

1.3 Equality of Matrices

The matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are said to be ***equal*** if $\mathbf{A}$ and $\mathbf{B}$ have the same size and $a_{ij} = b_{ij}$ for each i and j.

Example:

$$\begin{bmatrix} 2 & 3 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1+1 & 3 \\ 16/2 & 9-2 \end{bmatrix}$$

but

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 2 \end{bmatrix}$$

since the sizes of are not the same: $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is size 2×1 while $\begin{bmatrix} 2 & 2 \end{bmatrix}$ is size 1×2.

□

1.4 Transpose of a Matrix

Suppose $\mathbf{A}$ is an $m \times n$ matrix. The ***transpose*** of $\mathbf{A}$, denoted $\mathbf{A}^{\mathrm{T}}$, is the matrix of size $n \times m$ obtained by interchanging the rows and columns of $\mathbf{A}$. For example, if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

then

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 3 & -7 \\ 1 & 1 & -3 \end{bmatrix} .$$

Example: Let

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Big\} \text{ size } 2 \times 3$$

then

$$\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \Bigg\} \text{ size } 3 \times 2$$

□

Using this last example, notice that

$$(\mathbf{B}^{\mathrm{T}})^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \mathbf{B} .$$

This turns out to be true in general: for any matrix $\mathbf{A}$, $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$.

Example: Let

$$\mathbf{U} = \begin{bmatrix} 2 \\ -1/2 \\ \sqrt{2} \end{bmatrix} \quad \Bigg\} \text{ size } 3 \times 1$$

then

$$\mathbf{U}^{\mathrm{T}} = \begin{bmatrix} 1 & -1/2 & \sqrt{2} \end{bmatrix} \quad \} \text{ size } 1 \times 3$$

□

1.5 Some Special Matrices

Zero Matrix The $m \times n$ ***zero matrix*** is the $m \times n$ matrix with all entries zero. For example, the 2×3 zero matrix is

$$\mathbf{0}_{\mathbf{2\times 3}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Square Matrix A matrix with the same number n of rows and columns is called a ***square matrix*** of order n. For example,

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

Here the entries $a_{11} = 1$, $a_{22} = 3$, and $a_{33} = -3$ (reading from upper-left to lower-right) form the ***main diagonal*** of $\mathbf{A}$.

Diagonal Matrix A square matrix $[a_{ij}]_{n\times n}$ with all entries not on the main diagonal equal to zero is called a ***diagonal matrix***. That is, $a_{ij} = 0$ if $i \neq j$. For example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

is a diagonal matrix, however

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 9 & 0 & 0 \end{bmatrix}$$

is not.

Upper Triangular Matrix A square matrix is called ***upper triangular*** if all entries below the main diagonal are zero, for example

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -3 \end{bmatrix} .$$

Lower Triangular Matrix A square matrix is called ***lower triangular*** if all entries above the main diagonal are zero, for example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 5 & -7 & -3 \end{bmatrix} .$$

Triangular Matrix A matrix which is either upper or lower triangular.

2 Matrix Addition and Scalar Multiplication

Matrices inherit many of the properties of ordinary real numbers, including some of the operations of arithmetic. Here we define two of these operations: addition of matrices, and the multiplication of a matrix by a number, called scalar multiplication.

2.1 Matrix Addition

Let $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{m\times n}$. Then

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]_{m\times n} .$$

That is, provided $\mathbf{A}$ and $\mathbf{B}$ are the same size, the matrix $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is simply the $m \times n$ matrix formed by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$: $c_{ij} = a_{ij} + b_{ij}$.

Example: Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 3 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -11 & 1 & 12 \\ 7 & -2 & 13 \end{bmatrix}.$$

Then

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 3+(-11) & 0+1 & (-2)+12 \\ 4+7 & 3+(-2) & 1+13 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 10 \\ 11 & 1 & 14 \end{bmatrix}.$$

□

Example: Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ -2 & 4 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{0_{3\times 2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\mathbf{A} + \mathbf{0_{3\times 2}} = \mathbf{A}$ as one would expect.

□

This last example generalizes: matrices inherit the addition properties of the real numbers: for matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{0_{m\times n}}$ each of size $m \times n$:

(i) $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$.

(ii) $\mathbf{A}+(\mathbf{B}+\mathbf{C}) = (\mathbf{A}+\mathbf{B})+\mathbf{C}$.

(iii) $\mathbf{A}+\mathbf{0_{m\times n}} = \mathbf{A}$.

(iv) $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}+\mathbf{B}^{\mathrm{T}}$.

2.2 Scalar Multiplication

Let $\mathbf{A} = [a_{ij}]_{m\times n}$ be a matrix and let k be a real number (a ***scalar***). Then $\mathbf{C} = k\mathbf{A}$ is the matrix with entry $c_{ij} = ka_{ij}$. For example, for

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix},$$

multiplication by the scalar 1/2 gives

$$\frac{1}{2}\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} (1/2)(1) & (1/2)(6) \\ (1/2)(4) & (1/2)(2) \end{bmatrix} = \begin{bmatrix} 1/2 & 3 \\ 2 & 1 \end{bmatrix}.$$

□

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -3/2 \\ 7 \end{bmatrix}.$$

Then

$$\begin{aligned} 4\mathbf{A} + 2\mathbf{B} &= 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -3/2 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 12 \end{bmatrix} + \begin{bmatrix} -3 \\ 14 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 26 \end{bmatrix} . \end{aligned}$$

□

Like matrix addition, scalar multiplication inherits many of the multiplication rules of the ordinary real numbers. In the following, let k, k_1 and k_2 be scalars, and $\mathbf{A}$ and $\mathbf{B}$ be matrices of size $m \times n$:

(i) $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$.

(ii) $(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$.

(iii) $k_1(k_2\mathbf{A}) = (k_1k_2)\mathbf{A}$.

(iv) $0\mathbf{A} = \mathbf{0_{m\times n}}$.

(v) $k\mathbf{0_{m\times n}} = \mathbf{0_{m\times n}}$.

(vi) $(k\mathbf{A})^{\mathrm{T}} = k\mathbf{A}^{\mathrm{T}}$.

2.3 Matrix Subtraction

Now that we have clearly defined matrix addition and scalar multiplication, the operation of subtraction can be stated simply: if $\mathbf{A}$ and $\mathbf{B}$ are matrices of size $m \times n$, then we define

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} .$$

Here are a few examples to put this all together:

Example: Solve

$$3\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 12\begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix}$$

for $\begin{bmatrix} x \\ y \end{bmatrix}$.

Solution:

$$\begin{aligned}
3\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= 12\begin{bmatrix} -1/2 \\ 3/4 \end{bmatrix}, \\
3\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} -6 \\ 9 \end{bmatrix}, \\
3\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ 9 \end{bmatrix}, \\
3\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -5 \\ 11 \end{bmatrix}, \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{3}\begin{bmatrix} -5 \\ 11 \end{bmatrix}, \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -5/3 \\ 11/3 \end{bmatrix}.
\end{aligned}$$

Therefore $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/3 \\ 11/3 \end{bmatrix}$.

□

Example: Let

$$\mathbf{A} = \begin{bmatrix} 3 & -4 & 5 \\ -2 & 1 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 1 & 3 \\ 2 & 6 & -6 \end{bmatrix}.$$

Compute $2(3\mathbf{C} - \mathbf{A}) + 2\mathbf{B}$.

Solution:

First, notice that $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are all the same size, 2×3 in this case, so addition is defined. So we have

$$\begin{aligned}
&2\,(3\,\mathbf{C} - \mathbf{A}) + 2\,\mathbf{B} \\
=&2\left(3\begin{bmatrix} -1 & 1 & 3 \\ 2 & 6 & -6 \end{bmatrix} + (-1)\begin{bmatrix} 3 & -4 & 5 \\ -2 & 1 & 6 \end{bmatrix}\right) + 2\begin{bmatrix} 1 & 4 & 2 \\ 4 & 1 & 2 \end{bmatrix} \\
=&2\left(\begin{bmatrix} -3 & 3 & 9 \\ 6 & 18 & -18 \end{bmatrix} + \begin{bmatrix} -3 & 4 & -5 \\ 2 & -1 & -6 \end{bmatrix}\right) + \begin{bmatrix} 2 & 8 & 4 \\ 8 & 2 & 4 \end{bmatrix} \\
=&2\begin{bmatrix} -6 & 7 & 4 \\ 8 & 17 & -24 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 4 \\ 8 & 2 & 4 \end{bmatrix} \\
=&\begin{bmatrix} -12 & 14 & 8 \\ 16 & 34 & -48 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 4 \\ 8 & 2 & 4 \end{bmatrix} \\
=&\begin{bmatrix} 10 & 22 & 12 \\ 24 & 36 & -44 \end{bmatrix}
\end{aligned}$$

□

Problems for Sections 1 and 2

1. Let
$$\mathbf{Q} = \begin{bmatrix} 1/3 & 6 & -1/2 \\ 0 & -13 & 1/7 \\ 0 & 0 & -1 \end{bmatrix} .$$

   (i) State the size and order of $\mathbf{Q}$.

   (ii) State q_{21}, q_{13}, q_{33}.

   (iii) Is $\mathbf{Q}^{\mathrm{T}}$ upper triangular, lower triangular, or neither?

2. Construct $[a_{ij}]_{2\times 3}$ where $a_{ij} = -2i + 3j$.

3. Construct $[b_{ij}]_{3\times 3}$ where $b_{ij} = (-1)^{i+j}(i^2 + j^2)$.

4. A matrix $\mathbf{A}$ is called ***symmetric*** if $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$. Is the matrix of of Problem 3 symmetric?

5. Solve the matrix equation
$$\begin{bmatrix} 2x & 7 \\ 7 & 2y \end{bmatrix} = \begin{bmatrix} y & 7 \\ 7 & y \end{bmatrix} .$$

6. Find all x for which
$$\begin{bmatrix} x^2 + 2000x & \sqrt{x^2} \\ x^2 & \ln(e^x) \end{bmatrix} = \begin{bmatrix} 2001 & -x \\ 2001 - 2000x & x \end{bmatrix} .$$

7. Let
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -6 & -5 \\ 2 & -3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -2 & -1 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{0_{2\times 2}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

   Compute the following:

   (i) $2\mathbf{A} + 3(\mathbf{B} + \mathbf{C})$.

   (ii) $(1/2)\mathbf{A} - 2(\mathbf{B} + 2\mathbf{C})$.

   (iii) $(\mathbf{B} - 2\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}$.

   (iv) $-3(\mathbf{B} - 2\,\mathbf{0_{2\times 2}}) + 0\mathbf{B}$.

8. Solve for x and y:
$$x\begin{bmatrix} 3 \\ 2 \end{bmatrix} - y\begin{bmatrix} -4 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 4 \end{bmatrix} .$$

9. Solve for x and y:
$$3\begin{bmatrix} x \\ y \end{bmatrix} - 3\begin{bmatrix} -2 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 6 \\ -2 \end{bmatrix} .$$

10. Solve for x, y and z:
$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} x \\ y \\ 4z \end{bmatrix} = \begin{bmatrix} -10 \\ -24 \\ 14 \end{bmatrix} .$$

Answers to Problems for Sections 1 and 2

1. *(i)* size 3×3; order 3.

   (ii) $q_{21} = 0$, $q_{13} = -1/2$, $q_{33} = -1$.

   (iii) $\mathbf{Q}^{\mathrm{T}}$ is lower triangular.

2. $\begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -5 & 10 \\ -5 & 8 & -13 \\ 10 & -13 & 18 \end{bmatrix}$

4. Yes.

5. $x = y = 0$.

6. $x = -2001$.

7. *(i)* $\begin{bmatrix} -20 & -16 \\ 3 & -6 \end{bmatrix}$.

   (ii) $\begin{bmatrix} 21 & 29/2 \\ 19/2 & -15/2 \end{bmatrix}$.

   (iii) $\begin{bmatrix} -10 & 0 \\ -11 & 3 \end{bmatrix}$.

   (iv) $\begin{bmatrix} 18 & 15 \\ -6 & 9 \end{bmatrix}$.

8. $x = 90/29$, $y = -24/29$.

9. $x = 6$, $y = 4/3$.

10. $x = -6$, $y = -14$, $z = 1$.