1 The Binomial Theorem: Another Approach

1.1 Pascal's Triangle

In class (and in our text) we saw that, for integer $n \geq 1$, the binomial theorem can be stated

$$(a+b)^n = c_0 a^n + c_1 a^{n-1} b + c_2 a^{n-2} b^2 + \cdots + c_{n-1} a b^{n-1} + c_n b^n \ ,$$

where the coefficients $c_0, c_1, \ldots, c_n$ are given by the n^{th} row of Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

For example, to expand $(a+b)^5$ we would construct the triangle as above, and read off the coefficients from the fifth row to conclude:

$$(a+b)^5 = a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5 \ .$$

It would be nice if we could determine the coefficients $c_0, c_1, \ldots, c_n$ without having to construct the first n rows of the triangle. Fortunately, there is a way to do this... read on!

1.2 Factorial Notation and Binomial Coefficients

To obtain the coefficients in the expansion of $(a+b)^n$ for integer $n \geq 0$ without first constructing Pascal's triangle, we employ the ***factorial function***. For integer $n \geq 1$, define

$$n! = 1 \cdot 2 \cdot 3 \cdots n \ ,$$

and if $n = 0$ we define $0! = 1$. So, for example, $3! = 6$, since $3! = 3 \cdot 2 \cdot 1$. The expression $n!$ is read "n factorial" and this function arises in many areas of mathematics. $n!$ is the number of different ways of arranging n distinct objects, so it is useful in the study of probability and counting arguments.

We often have to simplify expressions involving factorials, as in

Example: Simplify

$$\frac{(n+3)!}{n!}$$

Solution:

$$\begin{aligned} \frac{(n+3)!}{n!} &= \frac{1 \cdot 2 \cdots n \cdot (n+1) \cdot (n+2) \cdot (n+3)}{1 \cdot 2 \cdots n} \\ &= (n+1) \cdot (n+2) \cdot (n+3) \end{aligned}$$

$\square$

Using factorial notation, we can now define the ***binomial coefficient*** $\binom{n}{r}$. For integer $n \geq 0$ and $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} .$$

The expression $\binom{n}{r}$, read "n choose r" is also commonly denoted ${}_nC_r$. Although we will not prove it here, one very important interpretation of $\binom{n}{r}$ (or ${}_nC_r$) is that it gives the number of different ways of forming a subset of r objects from a collection of n distinct objects.

Example: A lottery consists of randomly drawing six ping-pong balls from a collection of 49 numbered ping-pong balls. How many different outcomes are possible with this lottery?

Solution: We have 49 distinct objects and we are forming subsets of six, so there are

$$\begin{aligned}\binom{49}{6} &= \frac{49!}{43!6!} \\ &= \frac{44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\ &= 13{,}983{,}816 .\end{aligned}$$

So there are 13,983,816 possible outcomes for any particular draw. In other words, if we bought a single ticket with six numbers, the probability of our ticket matching the six numbers drawn is 1/13,983,816. □

The reason we are interested in the binomial coefficients is that these are precisely the numbers which appear in Pascal's triangle. That is, entry $(r+1)$ of row n of Pascal's triangle is $\binom{n}{r}$, so we may think of Pascal's triangle as

$$\begin{array}{lccccccccccc}
n=1: & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
n=2: & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
n=3: & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
n=4: & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
n=5: & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\vdots & & & & & & \vdots & & & & &
\end{array}$$

Using binomial coefficients, we may now restate the

Binomial Theorem: Let $n \geq 1$ be an integer. Then

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} b^n .$$

The binomial theorem in this form makes it much easier to answer questions such as

Example: What is the coefficient of x^{13} in the expansion of $(3x-5)^{20}$?

Solution: First, write

$$(3x-5)^{20} = [(3x)+(-5)]^{20}$$

so that, by the binomial theorem,

$$[(3x)+(-5)]^{20}$$
$$= \binom{20}{0}(3x)^{20} + \binom{20}{1}(3x)^{19}(-5) + \binom{20}{2}(3x)^{18}(-5)^2 + \cdots + \binom{20}{19}(3x)^1(-5)^{19} + \binom{20}{20}(-5)^{20} .$$

The x^{13} term is

$$\binom{20}{7}(3x)^{13}(-5)^7 = -\frac{20!}{13!7!}3^{13}5^7x^{13} ,$$

and so the coefficient of x^{13} in the expansion is

$$-\frac{20!}{13!7!}3^{13}5^7 = -9{,}655{,}618{,}668{,}750{,}000 .$$

Note that, in this case, computing $-9{,}655{,}618{,}668{,}750{,}000$ is no simple task, and so it is certainly acceptable (indeed preferable!) to state the coefficient in the form $-\frac{20!}{13!7!}3^{13}5^7$, or even as $-\binom{20}{13}3^{13}5^7$. □

1.3 Problems

1. Simplify $\frac{51!}{47!}$.

   ans: 5,997,600

2. For integer $n \geq 0$ and $0 \leq r \leq n$, simplify

$$\binom{n}{r} - \binom{n}{n-r}$$

   ans: 0

3. Expand $(\sqrt{x}+\sqrt{y})^4$.

   ans: $x^2 + 4x^{3/2}y^{1/2} + 6xy + 4x^{1/2}y^{3/2} + y^2$

4. Expand $(x^2-y^3)^5$.

   ans: $x^{10} - 5x^8y^3 + 10x^6y^6 - 10x^4y^9 + 5x^2y^{12} - y^{15}$

5. Simplify $\binom{9}{3}\binom{5}{2}$.

   ans: 840

6. Find the eleventh term in the expansion of $(2a - b^2)^{13}$.

ans: $2288a^3b^{20}$

7. A poker hand of five cards is dealt from a deck of 52 playing cards. How many different hands are possible?

ans: 2,598,960

8. Find the coefficient of x^n in the expansion of $(1 + x)^{2n}$.

ans: $(2n)!/(n!)^2$

9. Suppose $n \geq 1$ is an integer. The expression $\dfrac{(x+h)^n - x^n}{h}$ is not defined if $h = 0$. However, if you first simplify the expression (assuming $h \neq 0$) and then set $h = 0$, you get a very simple result. What is it?

ans: nx^{n-1}

10. If the coefficients of x^2 and x^5 are the same in the expansion of $(x + 1)^n$, what is n?

ans: $n = 7$

2 Sequences

2.1 Overview

A (numerical) ***sequence*** is a list of real numbers in which each entry is a function of its position in the list. The entries in the list are called ***terms***. For example,

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

is a sequence with first term 1, second term 1/2, third term 1/3, etc. A sequence is typically denoted $\{a_n\}_{n=1}^{\infty}$, where the ***subscript*** n, called the ***index***, indicates the position of the term a_n in the list. That is,

$$\{a_n\}_{n=1}^{\infty} = \underset{\substack{\uparrow \\ 1^{\text{st}} \\ \text{term}}}{a_1,} \quad \underset{\substack{\uparrow \\ 2^{\text{nd}} \\ \text{term}}}{a_2,} \quad \underset{\substack{\uparrow \\ 3^{\text{rd}} \\ \text{term}}}{a_3,} \quad \ldots$$

The terms of a sequence are often given as a formula, which gives us the "recipe" for the sequence. For example, the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ written out is

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Here, the general n^{th} term is $a_n = 1/n$, so $a_1 = 1/1$, $a_2 = 1/2$, and so on.

The index does not always start at $n = 1$. For that matter, the index need not be denoted with the letter n. For example,

$$\begin{aligned}\{2^k\}_{k=0}^{\infty} &= 2^0,\ 2^1,\ 2^2,\ 2^3,\ \ldots \\ &= 1,\ 2,\ 4,\ 8,\ \ldots\end{aligned}$$

Here, $a_k = 2^k$, $k \geq 0$.

Here's another example:

Example: Define a sequence by $b_k = k/(1 + 2^k)$, $k = 1, 2, 3, \ldots$. Write down the first three terms of the sequence.

Solution:

$$\begin{aligned}&b_1,\ b_2,\ b_3 \\ =&\frac{1}{1+2^1},\ \frac{2}{1+2^2},\ \frac{3}{1+2^3} \\ =&\frac{1}{3},\ \frac{2}{5},\ \frac{1}{3}\end{aligned}$$

□

We are interested in two specific types of sequences: ***(i) arithmetic*** and ***(ii) geometric***

2.2 Arithmetic Sequences

Definition: A sequence $a_1,\ a_2,\ a_3,\ \ldots$ with the property that

$$\begin{aligned}a_2 - a_1 &= d \\ a_3 - a_2 &= d \\ a_4 - a_3 &= d \\ &\vdots \\ a_n - a_{n-1} &= d \\ &\vdots\end{aligned}$$

is called an ***arithmetic sequence*** with ***common difference*** d.

In simple terms, an arithmetic sequence is characterized by the property that the difference between consecutive terms is the same. An arithmetic sequence is also called an ***arithmetic progression***.

Example: Let $a_n = 5 - 2n$, $n = 1, 2, 3, \ldots$

(i) List the first three terms of the sequence.

(ii) Is the sequence arithmetic?

(iii) If yes to *(ii)*, find the common difference.

Solution:

(i) The first three terms are

$$\begin{aligned} &a_1,\ a_2,\ a_3 \\ =&5-2(1),\ 5-2(2),\ 5-2(3) \\ =&3,\ 1,\ -1 \end{aligned}$$

(ii) Suppose $k \geq 1$ is any positive integer. Then

$$\begin{aligned} &a_{k+1}-a_k \\ =&[5-2(k+1)]-[5-2(k)] \\ =&5-2k-2-5+2k \\ =&-2 \end{aligned}$$

Since k was arbitrary, we conclude that the difference between any two consecutive terms is -2, and so the sequence is arithmetic.

(iii) From *(ii)* we conclude that the common difference is $d=-2$. □

Example: Suppose $\{a_n\}_{n=1}^{\infty}$ is an arithmetic sequence with first term 3 and common difference 7/3. Find a formula for a_n.

Solution: Since the common difference is 7/3 and the first term is 3, write out the first few terms to establish a pattern:

$$\begin{aligned} &a_1,\ a_2,\ a_3,\ a_4,\ \ldots \\ =&3,\ 3+7/3,\ 3+7/3+7/3,\ 3+7/3+7/3+7/3,\ \ldots \\ =&3,\ 3+7/3,\ 3+2(7/3),\ 3+3(7/3),\ \ldots \end{aligned}$$

By inspection, we see that term n has form $3+(n-1)(7/3)$. That is, $a_n=3+(n-1)(7/3)$. □

This last example generalizes to give a standard form for arithmetic sequences: an arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ with first term a and common difference d has n^{th} term $a_n=a+(n-1)d$.

2.3 Geometric Sequences

The general development of geometric sequences parallels that of arithmetic sequences, except that we consider division by a common value rather than addition:

Definition: A sequence $a_1,\ a_2,\ a_3,\ \ldots$ with the property that

$$\frac{a_2}{a_1} = r$$
$$\frac{a_3}{a_2} = r$$
$$\frac{a_4}{a_3} = r$$
$$\vdots$$
$$\frac{a_n}{a_{n-1}} = r$$
$$\vdots$$

is called a ***geometric sequence*** with ***common ratio*** r.

For a geometric sequence, the ratio of consecutive terms is the same. A geometric sequence is also called a ***geometric progression***.

Example: Let $\{b_n\}_{n=1}^{\infty} = \left\{\frac{3}{7^{n-1}}\right\}_{n=1}^{\infty}$.

(i) List the first three terms of the sequence.

(ii) Is the sequence geometric?

(iii) If yes to *(ii)*, find the common ratio.

Solution:

(i) The first three terms are

$$\begin{aligned} & b_1,\ b_2,\ b_3 \\ =& \frac{3}{7^{1-1}},\ \frac{3}{7^{2-1}},\ \frac{3}{7^{3-1}} \\ =& 3,\ \frac{3}{7},\ \frac{3}{49} \end{aligned}$$

(ii) Suppose $k \geq 1$ is any positive integer. Then

$$\begin{aligned} & \frac{b_{k+1}}{b_k} \\ =& \frac{3}{7^{k+1-1}} \Big/ \frac{3}{7^{k-1}} \\ =& \frac{3}{7^k}\frac{7^{k-1}}{3} \\ =& \frac{1}{7} \end{aligned}$$

Since k was arbitrary, we conclude that $b_{k+1}/b_k = 1/7$ for all integers $k \geq 1$, so that the sequence is geometric.

(iii) From *(ii)* we conclude that the common ratio is $r = 1/7$. □

Example: Suppose $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence with first term a and common ratio r. Find a formula for a_n.

Solution: Since the common ratio is r and the first term is a, the sequence has the form

$$\begin{aligned} & a_1,\ a_2,\ a_3,\ a_4,\ \ldots \\ =& a,\ a \cdot r,\ a \cdot r \cdot r,\ a \cdot r \cdot r \cdot r,\ \ldots \\ =& a,\ ar,\ ar^2,\ ar^3,\ \ldots \end{aligned}$$

By inspection, we see that term n has form ar^{n-1}. That is, $a_n = ar^{n-1}$. □

This last example leads us to conclude: a geometric sequence $\{a_n\}_{n=1}^{\infty}$ with first term a and common ratio r has n^{th} term $a_n = ar^{n-1}$.

2.4 Problems

1. Write the first four terms of the sequence defined by $a_n = \dfrac{2n-1}{n^2+2n}$, $n \geq 1$.

   ans: $1/3,\ 3/8,\ 1/3,\ 7/24$

2. For the sequence $\{a_n\}_{n=1}^{\infty}$ with first five terms $\sqrt{2}$, 2, $\sqrt{6}$, $2\sqrt{2}$, $\sqrt{10}$, give a possible expression for a_n.

   ans: $\sqrt{2n}$

3. Find an expression for a_n for the arithmetic sequence $3/5,\ 1/10,\ -2/5,\ \ldots$

   ans: $3/5 - (n-1)5/10$

4. Find the 14^{th} term of the arithmetic sequence $3,\ 7/3,\ 5/3,\ \ldots$

   ans: $-17/3$

5. An arithmetic sequence has $a_{17} = 25/3$ and $a_{32} = 95/6$. What is a_6?

   ans: $17/6$

6. If $a_1 = 25$, $d = -14$ and $a_n = -507$ then what is n if the sequence is arithmetic?

   ans: 39

7. Find the 23^{rd} term of the geometric sequence $7/625,\ -7/25,\ \ldots$

   ans: $7 \cdot 5^{40}$

8. Find an expression for a_n for the geometric sequence $2/x,\ 4/x^2,\ \ldots$

   ans: $(2/x)^n$

9. If a geometric sequence has $a_4 = 8/3$ and $a_7 = 64/3$, what is a_5?

ans: 16/3

10. A geometric sequence has the property that $a_{n+3} = 27a_n$. What then is r?

ans: 3