5 Inverse Matrices

5.1 Introduction

In our earlier work on matrix multiplication, we saw the idea of the inverse of a matrix. That is, for a square matrix $\mathbf{A}$, there may exist a matrix $\mathbf{B}$ with the property that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

This is a useful concept, and gives us yet another method for solving systems of equations. To illustrate, consider the simple system

$$\begin{aligned} 2x - 5y &= 6 \\ x + 3y &= 1 \ . \end{aligned}$$

Instead of writing this as an augmented matrix, write this as a matrix equation using a product:

$$\begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \ .$$

If we let $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, then the equation we wish to solve is

$$\mathbf{AX} = \mathbf{C} \ .$$

If we knew $\mathbf{A^{-1}}$, we could solve this easily for the unknown $\mathbf{X}$: (left) multiply both sides of the equation by $\mathbf{A^{-1}}$ to find

$$\begin{aligned} \mathbf{A^{-1}}(\mathbf{AX}) &= \mathbf{A^{-1}C} \\ (\mathbf{A^{-1}A})\mathbf{X} &= \mathbf{A^{-1}C} \\ \mathbf{IX} &= \mathbf{A^{-1}C} \\ \mathbf{X} &= \mathbf{A^{-1}C} \ . \end{aligned}$$

We see this is much like solving the simple equation $ax = c$ for the unknown x where a and c are real numbers.

In this section make precise the idea of a matrix inverse and develop a method to find the inverse of a given square matrix when it exists.

5.2 Definition

Suppose $\mathbf{A}$ is a square matrix of order n. A matrix $\mathbf{B}$ with the property that $\mathbf{BA} = \mathbf{I}$ is called an ***inverse*** of $\mathbf{A}$. If $\mathbf{A}$ has an inverse, it is called ***invertible***, and we write $\mathbf{A^{-1}}$ to denote the inverse.

Some notes concerning this definition:

1. If $\mathbf{A}$ is invertible, then $\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$.
2. If a matrix $\mathbf{A}$ has an inverse, then the inverse is unique, so we may speak of ***the*** inverse $\mathbf{A}$.
3. Not all square matrices have inverses.

5.3 Procedure for Finding the Inverse of a Matrix

Here we give a method for finding the inverse of a square matrix. We will see that this involves nothing more than row reduction that we have seen before. For the purposes of the explanation 2×2 matrices are used, but the method extends to square matrices of any size.

Suppose $\mathbf{A}$ is invertible, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and we wish to find a matrix $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ such that $\mathbf{AB} = \mathbf{I}$. That is, we want

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix multiplication may be expressed as two systems of equations:

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} &= 1 \\ a_{21}b_{11} + a_{22}b_{21} &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} a_{11}b_{12} + a_{12}b_{22} &= 0 \\ a_{21}b_{12} + a_{22}b_{22} &= 1 \end{aligned}$$

If $\mathbf{A}$ is invertible, then there are values of $b_{11}, b_{12}, b_{21}, b_{22}$ which solve this system. In augmented matrix form these two systems of equations become

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & 1 \\ a_{21} & a_{22} & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 1 \end{array}\right].$$

Now, if $\mathbf{A}$ is invertible, again meaning that these two systems have unique solutions, then after reduction by elementary row operations the result would be

$$\left[\begin{array}{cc|c} 1 & 0 & b_{11} \\ 0 & 1 & b_{21} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 1 & 0 & b_{12} \\ 0 & 1 & b_{22} \end{array}\right].$$

Here's the key observation: the elementary row operations used to reduce $\mathbf{A}$ are the same for both systems! Therefore, we can do both reductions simultaneously using an augmented matrix of the form

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{cc|cc} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{array}\right].$$

Notice what this says: if $\mathbf{A}^{-1}$ exists, then $\mathbf{A}$ reduces to $\mathbf{I}$ and produces $\mathbf{A}^{-1}$ in the augmented matrix above. It also tells us something more: if $\mathbf{A}$ fails to reduce to $\mathbf{I}$ with this procedure, then $\mathbf{A}^{-1}$ does not exist. So this procedure not only gives the inverse when it exists, it also tells us with certainty when $\mathbf{A}^{-1}$ does not exist.

The procedure can be summarized very concisely: to find the inverse of the matrix $\mathbf{A}$:

$$\left[\,\mathbf{A} \,\middle|\, \mathbf{I}\,\right] \xrightarrow{\text{reduce}} \left[\,\mathbf{I} \,\middle|\, \mathbf{A}^{-1}\,\right].$$

If the original matrix $\mathbf{A}$ does not reduce to $\mathbf{I}$ in this procedure, then $\mathbf{A}^{-1}$ does not exist.

5.4 Examples

Example: Back to our problem from the beginning of this section: solve the system

$$\begin{aligned} 2x - 5y &= 6 \\ x + 3y &= 1 \ . \end{aligned}$$

using matrix inverses.

Solution: Letting $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, we wish to solve

$$\mathbf{AX} = \mathbf{C} \ .$$

To find $\mathbf{A^{-1}}$, first set up

$$\left[\begin{array}{cc|cc} 2 & -5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \ .$$

Now reduce:

$$R_1 \leftrightarrow R_2 : \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 2 & -5 & 1 & 0 \end{array}\right]$$

$$(-2)R_1 + R_2 : \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -11 & 1 & -2 \end{array}\right]$$

$$(-1/11)R_2 : \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1/11 & 2/11 \end{array}\right]$$

$$(-3)R_2 + R_1 : \left[\begin{array}{cc|cc} 1 & 0 & 3/11 & 5/11 \\ 0 & 1 & -1/11 & 2/11 \end{array}\right]$$

Therefore, $\mathbf{A^{-1}} = \begin{bmatrix} 3/11 & 5/11 \\ -1/11 & 2/11 \end{bmatrix}$, and so

$$\begin{aligned} \mathbf{X} &= \mathbf{A^{-1}C} \\ &= \begin{bmatrix} 3/11 & 5/11 \\ -1/11 & 2/11 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 23/11 \\ -4/11 \end{bmatrix} \ . \end{aligned}$$

□

Example: Let

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{bmatrix} \ .$$

Find $\mathbf{A^{-1}}$.

Solution: Set up

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 5 & -7 & -3 & 0 & 0 & 1 \end{array}\right].$$

Now reduce:

$$\begin{array}{r} (2)R_1 + R_2 : \\ (-5)R_1 + R_3 : \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 2 & 1 & 0 \\ 0 & 3 & -8 & -5 & 0 & 1 \end{array}\right]$$

$$(-1)R_2 : \left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & -1 & 0 \\ 0 & 3 & -8 & -5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} (2)R_2 + R_1 : \\ (-3)R_2 + R_3 : \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & -5 & -3 & -2 & 0 \\ 0 & 1 & -3 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right]$$

$$\begin{array}{r} (5)R_3 + R_1 : \\ (3)R_3 + R_2 : \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 13 & 5 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \end{array}\right].$$

Since $\mathbf{A}$ reduced to $\mathbf{I}$ in the left hand side of the augmented matrix, the right hand side is $\mathbf{A^{-1}}$:

$$\mathbf{A^{-1}} = \left[\begin{array}{rrr} 2 & 13 & 5 \\ 1 & 8 & 3 \\ 1 & 3 & 1 \end{array}\right].$$

A check shows that indeed, $\mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$.

□

Example: Let

$$\mathbf{A} = \left[\begin{array}{rrr} 1 & 3 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right].$$

Find $\mathbf{A^{-1}}$.

Solution: Set up

$$\left[\begin{array}{rrr|rrr} 1 & 3 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Now reduce:

$$\begin{array}{r} (-2)R_1 + R_2 : \\ (-1)R_1 + R_3 : \end{array} \left[\begin{array}{rrr|rrr} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$(-1/5)R_2 : \left[\begin{array}{rrr|rrr} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2/5 & -1/5 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} (-3)R_2 + R_1 : \\ (2)R_2 + R_3 : \end{array} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1/5 & 3/5 & 0 \\ 0 & 1 & 1 & 2/5 & -1/5 & 0 \\ 0 & 0 & 0 & -1/5 & -2/5 & 1 \end{array}\right].$$

Notice: the left hand side of the augmented matrix is now reduced, but it is not the 3×3 identity matrix. Therefore, $\mathbf{A^{-1}}$ does not exist.

□

Example: Find $\mathbf{A^{-1}}$ if $\mathbf{A} = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$.

Solution:

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$(1/a)\, R_1 : \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \text{ assuming } a \neq 0$$

$$(-c)R_1 + R_2 : \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\left(\frac{a}{ad-bc}\right) R_2 : \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \text{ assuming } ad - bc \neq 0$$

$$\left(-\frac{b}{a}\right) R_2 + R_1 : \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right].$$

The conclusion is that if $\mathbf{A} = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ where $ad - bc \neq 0$, then $\mathbf{A^{-1}} = \frac{1}{ad-bc}\left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right]$. Note that even though we stated that $a \neq 0$ in the first row reduction step, the final result is valid even if $a = 0$. This form for $\mathbf{A^{-1}}$ is very convenient in practice.

□

Problems for Section 5

Find the inverses (if they exist) of the following matrices:

1. $$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

2. $$\begin{bmatrix} 6 & 9 \\ 4 & 6 \end{bmatrix}$$

3. $$\begin{bmatrix} 4 & -3 \\ 1 & -2 \end{bmatrix}$$

4. $$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 & -4 & 8 \\ 1 & -3 & 2 \\ 2 & -7 & 10 \end{bmatrix}$$

6. Use your answers to 1. and 3. to verify that $(\mathbf{AB})^{-1} = \mathbf{B^{-1}A^{-1}}$.

Solutions to Problems for Section 5

1. $$\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

2. Does not exist.

3. $$\begin{bmatrix} 2/5 & -3/5 \\ 1/5 & -4/5 \end{bmatrix}$$

4. $$\begin{bmatrix} 3/8 & -1/4 & 1/8 \\ -1/8 & 3/4 & -3/8 \\ -1/4 & 1/2 & 1/4 \end{bmatrix}$$

5. Does not exist.