4 Solving Systems of Equations by Reducing Matrices

4.1 Introduction

One of the main applications of matrix methods is the solution of ***systems of linear equations***. Consider for example solving the system

$$\begin{aligned} 2x - 3y &= 2 \\ -x + 2y &= 1\ . \end{aligned}$$

As we observed before, this system can easily be solved using the method of substitution. Another more systematic method is that of ***reduction of matrices***, the solution by a sequence of steps which can be carried out efficiently using matrices.

Before explaining the method in detail, the idea behind it is first illustrated by applying it to solve the simple system above. In the following, we list on the left the steps of the solution applied to the system of equations, and on the right we show how the corresponding steps can be written in matrix form.

First, state the problem in matrix form:

$$\begin{array}{cl} ① & 2x - 3y = 2 \\ ② & -x + 2y = 1 \end{array} \qquad \left[\begin{array}{rr|r} 2 & -3 & 2 \\ -1 & 2 & 1 \end{array}\right]\ .$$

Now multiply the first equation by 1/2 to make the coefficient of x one:

$$\begin{array}{cl} ① & x - \frac{3}{2}y = 1 \\ ② & -x + 2y = 1 \end{array} \qquad \left[\begin{array}{rr|r} 1 & \frac{-3}{2} & 1 \\ -1 & 2 & 1 \end{array}\right]\ .$$

Now add the first equation to the second:

$$\begin{array}{cl} ① & x - \frac{3}{2}y = 1 \\ ② & \frac{1}{2}y = 2 \end{array} \qquad \left[\begin{array}{rr|r} 1 & \frac{-3}{2} & 1 \\ 0 & \frac{1}{2} & 2 \end{array}\right]\ .$$

Now multiply the second equation by 2:

$$\begin{array}{cl} ① & x - \frac{3}{2}y = 1 \\ ② & y = 4 \end{array} \qquad \left[\begin{array}{rr|r} 1 & \frac{-3}{2} & 1 \\ 0 & 1 & 4 \end{array}\right]\ .$$

Add 3/2 times the second equation to the first:

$$\begin{array}{cl} ① & x = 7 \\ ② & y = 4 \end{array} \qquad \left[\begin{array}{rr|r} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]\ .$$

So we see that $x = 7$, $y = 4$ is the solution. Notice how the matrices at each step contain the same information as the equations while dispensing with the variables.

We say that the system of equations (or the matrix) at each step is ***equivalent*** to the one before it, meaning that it contains the same information. Although the system of equations looks different from step to step, each has the same solution $x = 7$, $y = 4$.

In the sections to follow we will describe in more detail the method of solving systems by reduction of matrices.

4.2 Some Terminology and Description of the Procedure

In this section we introduce some notation and terminology, describe the procedure used in the previous section, and give some examples illustrating the method.

4.2.1 Augmented Coefficient Matrix

With reference to the system from the previous section

$$\begin{array}{rcl} 2x - 3y & = & 2 \\ -x + 2y & = & 1 \end{array},$$

the matrix

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

is called the ***coefficient matrix***. The matrix

$$\left[\begin{array}{rr|r} 2 & -3 & 2 \\ -1 & 2 & 1 \end{array}\right]$$

is called the ***augmented coefficient matrix***.

4.2.2 Elementary Row Operations

The operations performed on the augmented matrix in the first section are called ***elementary row operations***. There are three types of elementary row operations:

(i) Interchanging two rows. The notation for this is $R_i \leftrightarrow R_j$, which means interchange rows i and j.

(ii) Multiplying a row by a nonzero constant. Notation: kR_i, meaning multiply row i by the constant k.

(iii) Adding k times one row to another. Notation: $kR_i + R_j$, meaning add k times row i to row j, leaving row i unchanged. With this notation, the row being changed (row j) is listed second.

Performing elementary row operation on the augmented matrix of a system of equation does not alter the information contained in the augmented matrix. That is, the system represented by the augmented matrix following the elementary row operation has the same solution as that before. This is an important point. If a matrix is obtained from another by one or more elementary row operations, the two matrices are said to be ***equivalent***.

It should be pointed out that the notation for the elementary row operations is not universal, and different books may use variations on the notation given here.

4.2.3 Reduced Form of a Matrix

The final augmented matrix in our problem had the form $\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]$; the $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ part of the augmented matrix is said to be in ***reduced form*** (or ***reduced row echelon form***). Although the reduced form in this case turned out to be the identity matrix of order two, this need not be the case. More generally, a matrix is said to be in ***reduced form*** if

(i) The first nonzero entry in a row (if any) is 1, while all other entries of the column containing that 1 are 0;

(ii) The first nonzero entry in a row is to the right of the first nonzero entry in each row above; and

(iii) Any rows consisting entirely of zeros are at the bottom of the matrix.

The "first nonzero entry in a row" is called the ***leading entry*** of the row.

Example: The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is not reduced, since the leading entry of the first row is 2, not 1. □

Example: The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is reduced since it meets the three conditions of being a reduced matrix. □

4.2.4 Procedure for Reducing a Matrix

We next give the procedure for reducing a matrix, followed by an example. The description of the procedure is slightly technical— it is best to work through several examples to understand how it works.

To Put a Matrix in Reduced Form: Suppose a matrix has n rows.

(i) Set $j = 1$ (here j is a "row counter", and so we are starting with the first row).

(ii) Rearrange rows $j, j+1, \ldots, n$ to that the leading entry of row j is positioned as far to the left as possible.

(iii) Multiply row j by a nonzero constant to make the leading entry equal 1.

(iv) Use this leading entry of 1 to reduce all other entries in its column to 0 using elementary row operations.

(v) If any of rows $j+1, \ldots, n$ contain nonzero terms, increase j by 1 and go to step *(ii)*

Here's an example illustrating the steps of this procedure along with the notation for the elementary row operation:

Example: Reduce the matrix

$$\begin{bmatrix} 0 & 4 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

Solution:

Rearrange rows 1 to 3 so that the leading entry of row 1 is as far to the left as possible. In this case, interchange rows 1 and 3. (We could instead interchange rows 1 and 2; either choice will work). This is step *(ii)* in the procedure above.

$R_1 \leftrightarrow R_3$: $$\begin{bmatrix} 2 & 0 & 3 \\ 2 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Multiply row 1 by 1/2 to make leading entry 1. This is step *(iii)* in the procedure above.

$(1/2)R_1$: $$\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 2 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Reduce all entries in the first column to zero using elementary row operations. In this case, -2 times row 1 is added to row 2. This is step *(iv)* in the procedure above.

$(-2)R_1 + R_2$: $$\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -1 & -3 \\ 0 & 4 & 1 \end{bmatrix}$$

Now on to step *(v)*, which is to examine the remaining rows (2 and 3 in this case) to see if any contain nonzero entries. Since the answer is yes, we return to step *(ii)*: rearrange the remaining rows 2 and 3 if necessary so that the leading entry of row 2 is as far to the left as possible. The leading term in each of rows 2 and 3 is in column 2, so no need to interchange rows. On to step *(iii)* then: multiply row 2 by -1 to make leading entry 1.

$(-1)R_2$: $\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

Now reduce all entries in the column containing the leading entry of row 2 (column 2 in this case): -4 times row 2 is added to row 3. This is step *(iv)* in the procedure.

$(-4)R_2 + R_3$: $\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{bmatrix}$

Now we're on to step *(v)* again: examine remaining rows to see if any nonzero entries remain. Since "yes", return to step *(ii)*. Only one row remains, row 3, so there are no rows to interchange. Multiply row 3 by $-1/11$ to make the leading entry 1. This is step *(iii)*.

$(-1/11)R_3$: $\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Using this new leading term of 1 in column 3, reduce the other entries in the column to 0. First, add -3 times row 3 to row 2.

$(-3)R_3 + R_2$: $\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Finally, add $-3/2$ times row 3 to row 1.

$(-3/2)R_3 + R_1$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So the reduced form of the original matrix $\begin{bmatrix} 0 & 4 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$ is the identity matrix of order 3.

□

4.2.5 An Example Illustrating the Method

Here's an example which puts together everything so far:

Example: Solve

$$\begin{aligned} x - 3y &= -11 \\ 4x + 3y &= 9 \end{aligned}$$

Solution: Writing the system as an augmented matrix:

$$\left[\begin{array}{rr|r} 1 & -3 & -11 \\ 4 & 3 & 9 \end{array}\right] .$$

Now reduce:

$$(-4)R_1 + R_2 : \left[\begin{array}{rr|r} 1 & -3 & -11 \\ 0 & 15 & 53 \end{array}\right]$$

$$(1/15)R_2 : \left[\begin{array}{rr|r} 1 & -3 & -11 \\ 0 & 1 & 53/15 \end{array}\right]$$

$$3R_2 + R_1 : \left[\begin{array}{rr|r} 1 & 0 & -2/5 \\ 0 & 1 & 53/15 \end{array}\right]$$

This last augmented matrix as a system of equations says

$$\begin{aligned} 1x + 0y &= -2/5 \\ 0x + 1y &= 53/15 , \end{aligned}$$

or in other words, $x = -2/5$, $y = 53/15$.

□

4.3 More Examples

Example: Solve

$$\begin{aligned} x - 3y &= 2 \\ -x + 3y &= 0 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{rr|r} 1 & -3 & 2 \\ -1 & 3 & 0 \end{array}\right] .$$

Now reduce:

$$R_1 + R_2 : \left[\begin{array}{rr|r} 1 & -3 & 2 \\ 0 & 0 & 2 \end{array}\right]$$

This augmented matrix says

$$\begin{aligned} 1x - 3y &= 2 \\ 0x + 0y &= 2 . \end{aligned}$$

Look at the second of these equations: $0x + 0y = 2$; this can never be satisfied for any x and y, and so the original system of equations

$$\begin{aligned} x - 3y &= 2 \\ -x + 3y &= 0 \end{aligned}$$

has no solution. This generalizes: if at any stage of reduction of the augmented matrix we have a row whose only nonzero term is in the last column, then the system has no solution.

□

Example: Solve

$$\begin{aligned} x + 3y + 2z &= 1 \\ x + y + 5z &= 10 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 1 & 5 & 10 \end{array}\right].$$

Now reduce:

$$(-1)R_1 + R_2 : \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & 3 & 9 \end{array}\right]$$

$$(-1/2)R_2 : \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -3/2 & -9/2 \end{array}\right]$$

$$(-3)R_2 + R_1 : \left[\begin{array}{ccc|c} 1 & 0 & 13/2 & 29/2 \\ 0 & 1 & -3/2 & -9/2 \end{array}\right]$$

Writing this as equations,

$$\begin{aligned} x + (13/2)z &= 29/2 \\ y - (3/2)z &= -9/2\ . \end{aligned}$$

In this case we see there are not enough equations to determine a single solution to the system. Once a value is assigned to any one of the variables, the other two are determined, and there are in fact infinitely many solutions to the system. This infinite family of solutions is stated using a ***parameter***. That is, let r be any real number. Then if $z = r$, then

$$y = \frac{-9}{2} + \frac{3}{2}z = \frac{3r - 9}{2}$$

and

$$x = \frac{29}{2} - \frac{13}{2}z = \frac{29 - 13r}{2}\ .$$

and so the solution is

$$x = \frac{29 - 13r}{2}, \qquad y = \frac{3r - 9}{2}, \qquad z = r$$

where r is any real number. This is called a ***parametric solution*** with parameter r.

□

Example: Solve

$$\begin{aligned} x_1 + 3x_2 &= 6 \\ 2x_1 + x_2 &= 7 \\ x_1 + x_2 &= 4 \end{aligned}$$

Solution: The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & 7 \\ 1 & 1 & 4 \end{array}\right] .$$

Now for the reduction. In this example we carry out multiple elementary row operations in some steps:

$$\begin{array}{r} (-2)R_1 + R_2 : \\ (-1)R_1 + R_3 : \end{array} \left[\begin{array}{rr|r} 1 & 3 & 6 \\ 0 & -5 & -5 \\ 0 & -2 & -2 \end{array}\right]$$

$$(-1/5)R_2 : \left[\begin{array}{rr|r} 1 & 3 & 6 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{array}\right]$$

$$\begin{array}{r} 2R_2 + R_3 : \\ (-3)R_2 + R_1 : \end{array} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, $x_1 = 3$ and $x_2 = 1$.

□

Example: Solve

$$\begin{aligned} -3z + x &= 2 + y \\ x &= 1 + z - y \\ 2x - 7/2 &= 5z + y \end{aligned}$$

Solution: First, put the system into a structured form so we can read off the coefficients:

$$\begin{aligned} x - y - 3z &= 2 \\ x + y - z &= 1 \\ 2x - y - 5z &= 7/2 \ . \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & -1 & -3 & 2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & -5 & 7/2 \end{array}\right] .$$

Now reduce:

$$\begin{array}{r} (-1)R_1 + R_2 : \\ (-2)R_1 + R_3 : \end{array} \left[\begin{array}{rrr|r} 1 & -1 & -3 & 2 \\ 0 & 2 & 2 & -1 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$$

$$(1/2)R_2 : \left[\begin{array}{rrr|r} 1 & -1 & -3 & 2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right]$$

$$\begin{array}{r} R_2 + R_1 : \\ (-1)R_2 + R_3 : \end{array} \left[\begin{array}{rrr|r} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In this example we again find fewer equations than unknowns in the reduced augmented matrix, and so there are an infinite number of solutions. If $z = r$, then

$$y = -\frac{1}{2} - z = -\frac{1}{2} - r$$

and

$$x = \frac{3}{2} + 2z = \frac{3}{2} + 2r\ .$$

The solution is therefore

$$x = \frac{3}{2} + 2r, \qquad y = -\frac{1}{2} - r, \qquad z = r$$

where r is any real number.

□

4.4 Summary

To conclude, we state in general terms the method of solving systems of equations by reducing matrices, and summarize the cases which may arise as solutions:

Summary: Solving Systems of Equations by Reducing Matrices To solve a system of m equations in n unknowns

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & c_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & c_m \end{array}$$

1. Set up the augmented coefficient matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{array}\right]$$

2. Put the coefficient matrix part

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right]$$

   of the augmented matrix into reduced form using elementary row operations.

3. State a conclusion:

   case (i) If any row of the reduced augmented matrix has a nonzero entry in the last column and zeros elsewhere, then there is no solution.

   case (ii) If the reduced augmented matrix has fewer nonzero rows than unknowns, then there are infinitely many solutions which may be stated using one or more parameters.

   case (iii) The system has a unique solution which can be read from the reduced augmented matrix.

Problems for Section 4

Solve the following systems of equations using matrix reduction:

1.

$$\begin{aligned} 4x + 2y &= 11 \\ 9x - 3y &= 6 \end{aligned}$$

2.

$$\begin{aligned} 10x - 4y &= -6 \\ 2x + 5y &= -24 \end{aligned}$$

3.

$$\begin{aligned} 9x + 12y &= 21 \\ -5x + 2y &= 10 \end{aligned}$$

4.

$$\begin{aligned} 3x - 9y &= 24 \\ -2x + 6y &= 3 \end{aligned}$$

5.

$$\begin{aligned} 8x - 2y - 6z &= 2 \\ 8x + y - z &= 5 \\ 4x + 2y + 4z &= 10 \end{aligned}$$

6.

$$\begin{aligned} 2x - 4y + 6z &= -8 \\ 6x + 2y - 2z &= 0 \\ 2x + 3y - 5z &= 1 \end{aligned}$$

7.

$$\begin{aligned} 2x - 4y - 3z &= 3 \\ x + 3y + z &= -1 \\ 10x + 2y - 4z &= 4 \end{aligned}$$

8.

$$\begin{aligned} 2x + 2y + 2z &= 2 \\ x + 2y + 3z &= 4 \\ 4x + 5y + 6z &= 7 \end{aligned}$$

Solutions to Problems for Section 4

1. $x = 3/2, y = 5/2$
2. $x = -63/29, y = -114/29$
3. $x = -1, y = 5/2$
4. no solution
5. $x = 3/2, y = -4, z = 3$
6. $x = -1, y = 6, z = 3$
7. $x = (r + 1)/2, y = (-r - 1)/2, z = r$, where r is any real number
8. $x = r - 2, y = 3 - 2r, z = r$, where r is any real number