Note: Structure your proofs by stating the full "**Proposition:**" followed by "**Proof:**" as in class, and please start each new proof on a new page. Indent your argument where appropriate for readability and be careful with notation. Keep your work neat and tidy, use complete sentences and correct punctuation, and please staple your assignment. Do not use pages torn from a coil notebook, or if you do, please trim off the ragged edge.

As with any proof, clarity of presentation is as important as solving the problem. Strive to make your proofs clear, concise and precise. Feel free to use results from earlier problems in subsequent ones.

Finally, proofs for many of the problems in this course are likely available online if one looks hard enough, but the point of this course is to learn analysis, and in particular how to write analysis proofs. As such, please resist turning to the internet for help except as a last resort. Instead, discuss solutions among your peers or see me for hints.

Late Assignment Policy: Assignments handed in t days late will incur a penalty of 4^t %. Here t is rounded up to the nearest integer.

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1. Give an example of a collection of sets $\{A_i : i \in \mathbb{N}\}$ with the property that each A_i is infinite, $A_i \cap A_j$ is infinite for each $i, j \in \mathbb{N}$, and $\cap_{i=1}^{\infty} A_i$ is nonempty and finite.

2. We've used the following "obvious" result in class, so let's prove it: Prove that if $A \subset B$ where B is finite, then A is finite. That is, use the assumed bijection $f : B \to \{1, \ldots, n\}$ to construct a bijection $g : A \to \{1, \ldots, m\}$.

3. Let $\mathbb{Q}[x]$ be the set of all polynomials in x with coefficients from $\mathbb{Q}$. That is,
$$\mathbb{Q}[x] = \left\{ \sum_{k=0}^{n-1} a_k x^k \ : \ a_k \in \mathbb{Q} \text{ and } n \in \mathbb{N} \right\} ,$$
(here we define $x^0 = 1$) . Show that $\mathbb{Q}[x]$ is countable.

4. An element $a \in \mathbb{R}$ is called **algebraic** over $\mathbb{Q}$ if $p(a) = 0$ for some non-zero $p \in \mathbb{Q}[x]$. For example, every $q \in \mathbb{Q}$ is algebraic over $\mathbb{Q}$ (why?), as is $\sqrt{2}$ (since $p(\sqrt{2}) = 0$ where $p(x) = x^2 - 2 \in \mathbb{Q}[x]$), but neither $\pi = 3.14159...$ nor $e = 2.71828...$ are algebraic over $\mathbb{Q}$ (harder to show.) Real numbers that are not algebraic over $\mathbb{Q}$ are said to be **transcendental** over $\mathbb{Q}$.

   Show that the set of elements of $\mathbb{R}$ which are algebraic over $\mathbb{Q}$ is countably infinite. (Here you may assume the following corollary of the Fundamental Theorem of Algebra: if $p \in \mathbb{Q}[x]$ has degree n then p has at most n real roots.)

5. In the proof that the cardinality of a set is strictly less than that of its power set we considered mappings $f : A \to \mathcal{P}(A)$ and sets of the form $\{x \in A : x \notin f(x)\}$.

   (i) Give an example of a mapping $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ with the property that $\{n \in \mathbb{N} : n \notin f(n)\} = \emptyset$.

   (ii) Give an example of a mapping $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ with the property that $\{n \in \mathbb{N} : n \notin g(n)\} = \mathbb{N}$.

6. Soon it will be useful to have examples of real numbers that are not rational. Prove that for $p \in \mathbb{N}$ a prime number and $n \geq 2$ a natural number, $p^{1/n} \notin \mathbb{Q}$. That is, prove that all of the n^{th} roots of a prime number are irrational.

7. (A bit tricky) Let $A = [0, 1]$ and $B = [0, 1)$. Prove that $|A| = |B|$ by finding an explicit bijection $f : A \to B$.