Math 371 - Introductory Real Analysis

Sep 20 2019

Ordered Sets
&
the Real Numbers

Ordered Sets

- **Definition:** A is an ordered set if there exists a relation "$<$" such that
  (i) For any $x \in A$ and $y \in A$ exactly one of
  $$x < y, \quad x = y, \quad y < x$$
  is true.

  (ii) If $x < y$ and $y < z$ then $x < z$

  (iii) $\leq$, $>$, $\geq$ have the standard meaning.

- **Examples:** $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are ordered sets, but $\mathbb{C}$ is not, nor is $(\mathbb{Z}/n\mathbb{Z})$.

Bounded Sets: Definitions

Let $E \subset A$ where A is an ordered set.

- **Definition:** If there is $b \in A$ such that $x \leq b$ for every $x \in E$ we say that E is bounded above and b is an upper bound for E.
- **Definition:** If b_0 is an upper bound for E and $b_0 \leq b$ for every other upper bound b, then b_0 is called the least upper bound of E or the supremum of E, and we write

$$b_0 = \sup E, \qquad \text{read "soup of } E\text{"}$$

- **Definition:** If there is $a \in A$ such that $x \geq a$ for every $x \in E$ we say that E is bounded below and a is a lower bound for E.
- **Definition:** If a_0 is a lower bound for E and $a_0 \geq a$ for every other lower bound a, then a_0 is called the greatest lower bound of E or the infimum of E, and we write

$$a_0 = \inf E, \qquad \text{read "inf of } E\text{"}$$

Bounded Sets: Examples

- **Example:** $E = \{2, 3, 4\} \subset \mathbb{N}$.

  1 is a lower bound for E, as is 2. 10 is an upper bound for E, as is 1000.

  But $\inf E = 2$, $\sup E = 4$

- **Example:** $E = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset \mathbb{Q}$.

  $\inf E = 0 \notin E$, $\sup E = 1 \in E$

- **Example:** $E = \left\{\sum_{k=0}^{n} \frac{1}{k!} \mid n \in \mathbb{N}\right\} \subset \mathbb{Q}$.

  $\inf E = 1 \in E$, $\sup E$ does not exist in $\mathbb{Q}$ ($\sup E = e$ in fact).

Least Upper Bound Property

- **Definition:** An ordered set A has the least upper bound property if every nonempty subset $E \subset A$ that is bounded above has a least upper bound in A.

  That is, $\sup E$ exists and $\sup E \in A$

- **Example:** We saw that $\mathbb{Q}$ does not have the least upper bound property since $e = 2.71828\cdots = \sup\left\{\sum_{k=0}^{n} \frac{1}{k!} \mid n \in \mathbb{N}\right\} \notin \mathbb{Q}$.

- To handle limits we need to extend $\mathbb{Q}$ to a field which has the least upper bound property.

Fields

Definition: A field is a set F together with two operations $+$ and $\cdot$ such that for any $x, y, z \in F$:

1. $x + y \in F$
2. $x + y = y + x$
3. $(x + y) + z = x + (y + z)$
4. There exists a zero element $0 \in F$ such that $0 + x = x$
5. There exists an element $-x$ such that $x + (-x) = 0$
6. $x \cdot y \in F$
7. $x \cdot y = y \cdot x$
8. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
9. There exists a unit element $1 \in F$ such that $1 \cdot x = x$
10. If $x \neq 0$ there exists an element $1/x$ such that $(1/x) \cdot x = 1$
11. $x \cdot (y + z) = x \cdot y + x \cdot z$
12. $1 \neq 0$

Examples of Fields

- Familiar: $(\mathbb{Q}, +, \cdot)$ is a field
- More unusual: Recall that for $a, p \in \mathbb{N}$,
  $a \mod p =$ remainder upon division of a by p

  Let p be a prime number and $\mathbb{F} = \{1, 2, \ldots, p-1\}$.

  For $a, b \in \mathbb{F}$ define $a +_{\mathbb{F}} b = a + b \mod p$

  define $a \cdot_{\mathbb{F}} b = ab \mod p$

  Then $(\mathbb{F}, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$ is a field

Ordered Fields

- **Definition:** An ordered set F is an ordered field if
  - F is a field (satisfies the field axioms),
  - $x < y \implies x + z < y + z$
  - $x > 0$ and $y > 0 \implies xy > 0$
- $(\mathbb{Q}, +, \cdot)$ is an ordered field, but $(\mathbb{F}, +_{\mathbb{F}}, \cdot_{\mathbb{F}})$ is not.

Ordered Fields

The usual notions of positive ($x > 0$) and negative ($x < 0$) are defined for ordered fields, and the familiar operations and results involving inequalities still hold:

Proposition: For $x, y, z \in F$ an ordered field,

- $x > 0 \implies -x < 0$
- $x > 0$ and $y < z \implies xy < xz$
- $x < 0$ and $y < z \implies xy > xz$
- $x \neq 0 \implies x^2 > 0$
- $0 < x < y \implies 0 < 1/y < 1/x$

The Real Numbers

- **Theorem:** There exists a unique ordered field $\mathbb{R}$ with the least upper bound property such that $\mathbb{Q} \subset \mathbb{R}$
- **Note:** There are several techniques for constructing $\mathbb{R}$. Two of the more popular are construction using Cauchy sequences, and construction using Dedeking cuts.
- In summary:
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$
where $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, but only $\mathbb{R}$ has the least upper bound property.
- $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are countably infinite, but $\mathbb{R}$ is uncountable.
- The set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.